The Laplacian Operator in Polar Coordinates

Our goal is to study the heat, wave and Laplace's equation in (1) polar coordinates in the plane and (2) cylindrical coordinates in space. Here we derive the form of the Laplacian operator

$$\Delta u = u_{xx} + u_{yy} \tag{1}$$

in polar coordinates.

Recall that the transformation equations relating Cartesian coordinates (x, y) and polar coordinates (r, θ) are:

$$x = r\cos(\theta) \tag{2}$$
$$y = r\sin(\theta) \tag{3}$$

To transform the Laplacian operator we need to use the chain rule. We first compute the derivatives of x and y with respect to the polar variables r and θ:

$$\frac{\partial x}{\partial r} = \cos(\theta) \quad , \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta) \tag{4}$$
$$\frac{\partial y}{\partial r} = \sin(\theta) \quad , \quad \frac{\partial y}{\partial \theta} = r\cos(\theta) \tag{5}$$

Now, given $u(x, y)$ we consider its polar coordinate form $\bar{u}(r, \theta) = u(r\cos(\theta), r\sin(\theta))$. We first compute the first derivatives of $\bar{u}$ with respect to r and θ.

$$\frac{\partial \bar{u}}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

Similarly

$$\frac{\partial \bar{u}}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin(\theta) + r\frac{\partial u}{\partial y}\cos(\theta)$$

Hence

$$\boxed{\frac{\partial \bar{u}}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)} \tag{6}$$

$$\boxed{\frac{\partial \bar{u}}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin(\theta) + r\frac{\partial u}{\partial y}\cos(\theta)} \tag{7}$$

Next we need to compute the formulas for the second derivatives.

$$\begin{aligned}
\frac{\partial^2 \bar{u}}{\partial r^2} &= \frac{\partial}{\partial r}\Big(\frac{\partial \bar{u}}{\partial r}\Big) \\
&= \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)\Big) \\
&= \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial x}\Big)\cos(\theta) + \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y}\Big)\sin(\theta) \\
&= \Big(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial y}{\partial r}\Big)\cos(\theta) + \Big(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}\Big)\sin(\theta) \\
&= \Big(\frac{\partial^2 u}{\partial x^2}\cos(\theta) + \frac{\partial^2 u}{\partial y \partial x}\sin(\theta)\Big)\cos(\theta) + \Big(\frac{\partial^2 u}{\partial x \partial y}\cos(\theta) + \frac{\partial^2 u}{\partial y^2}\sin(\theta)\Big)\sin(\theta) \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 u}{\partial y^2}\sin^2(\theta) + 2\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta)
\end{aligned}$$

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Hence

$$\boxed{\frac{\partial^2 \bar{u}}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 u}{\partial y^2}\sin^2(\theta) + 2\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta)} \tag{8}$$

A similar, but more complicated calculation leads to the formula for $\dfrac{\partial^2 \bar{u}}{\partial \theta^2}$:

$$\boxed{\frac{\partial^2 \bar{u}}{\partial \theta^2} = r^2\frac{\partial^2 u}{\partial x^2}\sin^2(\theta) + r\frac{\partial^2 u}{\partial y^2}\cos^2(\theta) - r\Big(\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)\Big)} \tag{9}$$

Here are the details of the calculation:

$$\begin{aligned}
\frac{\partial^2 \bar{u}}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\Big(\frac{\partial \bar{u}}{\partial \theta}\Big) \\
&= \frac{\partial}{\partial \theta}\Big(- r\frac{\partial u}{\partial x}\sin(\theta) + r\frac{\partial u}{\partial y}\cos(\theta)\Big) \\
&= -r\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial x}\Big)\sin(\theta) - r\Big(\frac{\partial u}{\partial x}\Big)\cos(\theta) + r\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial y}\Big)\cos(\theta) - r\frac{\partial u}{\partial y}\sin(\theta)\Big) \\
&= -r\Big(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y \partial x}\frac{\partial y}{\partial \theta}\Big)\sin(\theta) - r\Big(\frac{\partial u}{\partial x}\Big)\cos(\theta) + r\Big(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}\Big)\cos(\theta) - r\frac{\partial u}{\partial y}\sin(\theta)\Big) \\
&= -r\Big(\frac{\partial^2 u}{\partial x^2}(-r\sin(\theta)) + \frac{\partial^2 u}{\partial y \partial x}(r\cos(\theta))\Big)\sin(\theta) - r\Big(\frac{\partial u}{\partial x}\Big)\cos(\theta) \\
&\quad + r\Big(\frac{\partial^2 u}{\partial x \partial y}(-r\sin(\theta)) + \frac{\partial^2 u}{\partial y^2}\frac{\partial x}{\partial \theta}\Big)\cos(\theta) - r\frac{\partial u}{\partial y}\sin(\theta)\Big) \\
&= r^2\frac{\partial^2 u}{\partial x^2}\sin^2(\theta) + r^2\frac{\partial^2 u}{\partial y^2}\cos^2(\theta) - 2r^2\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta) - r\Big(\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)\Big)
\end{aligned}$$

We bring together the 4 boxed formulas to see how to finish this calculation.

$$\boxed{\frac{\partial \bar{u}}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)} \tag{10}$$

$$\boxed{\frac{\partial \bar{u}}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin(\theta) + r\frac{\partial u}{\partial y}\cos(\theta)} \tag{11}$$

$$\boxed{\frac{\partial^2 \bar{u}}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 u}{\partial y^2}\sin^2(\theta) + 2\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta)} \tag{12}$$

$$\boxed{\frac{\partial^2 \bar{u}}{\partial \theta^2} = r^2\frac{\partial^2 u}{\partial x^2}\sin^2(\theta) + r^2\frac{\partial^2 u}{\partial y^2}\cos^2(\theta) - 2r^2\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta) - r\Big(\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)\Big)} \tag{13}$$

Notice first that the last term in equation (13) is, by (10), equal to $-r\dfrac{\partial \bar{u}}{\partial r}$, so we will write that way in a moment. Next compare the first two terms involving $\dfrac{\partial^2 u}{\partial x^2}$ in equations (12) and (13). Except for the r^2 factor in equation (13), those two terms would add together to give $\dfrac{\partial^2 u}{\partial x^2}$. Similarly for the terms involving $\dfrac{\partial^2 u}{\partial y^2}$ in the same equations. So, let's multiply (12) by r^2 and add the result to equation (13) to get

$$\begin{aligned}
r^2\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{\partial^2 \bar{u}}{\partial \theta^2} &= r^2\Big(\frac{\partial^2 u}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 u}{\partial y^2}\sin^2(\theta) + 2\cancel{\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta)}\Big) \\
&\quad + \Big(r^2\frac{\partial^2 u}{\partial x^2}\sin^2(\theta) + r^2\frac{\partial^2 u}{\partial y^2}\cos^2(\theta) - 2r^2\cancel{\frac{\partial^2 u}{\partial x \partial y}\cos(\theta)\sin(\theta)} - r\frac{\partial \bar{u}}{\partial r}\Big) \\
&= r^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) - r\frac{\partial \bar{u}}{\partial r}
\end{aligned}$$

Moving the last term on the right-hand side to the left side, and dividing both sides by r^2 we obtain

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{14}$$

Hence the Laplacian operator is Polar Coordinates in the plane is

$$\boxed{\Delta u = \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{u}}{\partial \theta^2}} \tag{15}$$

