A Geometrical Model of Lepton Decay Based on Time Quantization

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Abstract

The unphysical runaway solutions of the Lorentz-Dirac equation are reanalyzed using the local Fermi-Walker rest frames of particles. We show that the radiation reaction term can be removed by a quantization of time ansatz. The result is that the runaway solutions for the free electron can be eliminated, and the runaway solutions for particles with masses greater than the electron can be reinterpreted as relating to particle decay processes. We elevate the coupling constant τ that appears in the Lorentz-Dirac equation to a classical parameter alongside the parameters mass m and charge q, and then geometrize the theory by introducing an $SO(1,2)$ geometry on the classical parameter space of triples (τ, m, q). For processes with $\Delta q = 0$ we show that the eigenstates of the $SO(1,2)$ parameter space metric describe massive leptons and their corresponding neutrinos. We present a toy dynamical model that predicts masses close to the experimental masses of the tauon, the pion and the muon. When compared with the standard electroweak model we find that the electroweak coupling constants g_l are defined by $g_l = \frac{\tau_H}{\tau_l}$ where τ_H and τ_l are the new time parameters of the Higgs particle and lepton l, respectively.

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1 Introduction

The classical theory of the electron as developed by Lorentz, Poincarè and others around the turn of the century was given a firm theoretical foundation in Dirac's 1938 derivation (Dirac, 1938) of the Lorentz-Dirac equation

$$a^i = \frac{q}{mc} F_{(ext)}{}^i{}_j u^j + \frac{2q^2}{3mc^3}(\dot{a}^i - \frac{a^2}{c^2} u^i) \quad . \tag{1}$$

Dirac's derivation of this equation was based on the conservation law of energy-momentum for particles and fields, and his approach led in a Lorentz covariant way to the radiation reaction term $\frac{2q^2}{3mc^3}(\dot{a}^i - \frac{a^2}{c^2}u^i)$ that occurs on the right in (1). This term must be added to the Lorentz force term in (1) in order to account for the back reaction on the electron as it radiates electromagnetic energy when it is accelerated by external forces.

Although the Lorentz-Dirac theory successfully accounts for the radiation losses from accelerated charged particles, it has a number of unpleasant features that have kept the theory from being completely successful, and today the theory is generally regarded as only an approximation to the correct quantum theory of the electron. Included among the unpleasant features are the appearance of third time derivatives, the self-accelerating ("runaway") solutions for the free charged particle, and the acausal "pre-accelerations" associated with forced motion. These features are all roughly characterized by the coupling constant

$$\tau = \frac{2q^2}{3mc^3} \tag{2}$$

that occurs in the Lorentz-Dirac equation (1). Since this parameter has the value 0.62×10^{-23} seconds for an electron, it is usually argued that the non-intuitive affects mentioned above can be disregarded in a classical theory as they occur over a time scale that is well outside the classical domain. Others have argued that equation (1) is incomplete as it stands, and should be replaced by an equation that incorporates asymptotic boundary conditions. Rohrlich (Rohrlich, 1965) has carried through this program by replacing (1) with an integro-differential equation that successfully eliminates the runaway self-accelerating solutions for free charged particles. However, the resulting equation is non-local in time.

One might conclude that for the Lorentz-Dirac theory to fit into a classical picture of charged particles and electromagnetic fields so as to agree as closely as possible with experimental results, then something like Rohrlich's formulation of the theory is inevitable. On the other hand the Lorentz-Dirac theory based on equation (1) is such a beautiful theory, founded as it is on the general principle of conservation of energy-momentum, that one is led to ask if there is not an alternative interpretation of the physics associated with the Lorentz-Dirac equation that is also in harmony with the classical physical world. In this paper we take a fresh look at the Lorentz-Dirac theory with this question in mind.

Over the last approximately 100 years during which the classical theory of the electron has been developed, other advances have been made in understanding electromagnetic phenomena. In particular it is now well established that electromagnetism is just one part of a larger theory that unifies the electromagnetic interaction with the weak interaction. Indeed, the electroweak theory of Glashow (Glashow, 1980), Salam (Salam, 1980) and Weinberg (Weinberg, 1980) is today the standard model of the unified theory of the electromagnetic and weak interactions. In this paper we show that the radiation reaction term in the Lorentz-Dirac equation (1) can be viewed as a "remnant" of an electroweak interaction in that the term will be shown to lead to a heuristic model for the decay of charged particles in processes of the type

$$\pi^- \longrightarrow l^- + \bar{\nu}_l \tag{3}$$

$$l^- \longrightarrow l_1^- + \bar{\nu}_1 + \nu \tag{4}$$

These processes are characterized energetically by the well-known formulas (Jackson, 1967)

$$E_+ = \frac{m^2 + m_1^2}{2m} c^2 \tag{5}$$

$$E_0 = \frac{m^2 - m_1^2}{2m} c^2 \tag{6}$$

which follow from the principle of conservation of relativistic energy-momentum. If the decay process (3) occurs in the rest frame of the pion, then E_+ and E_0 represent the exact total energies, respectively, of the lepton ($m_1 = m_l > 0$) and the massless antineutrino ($m_0 = m_{\bar{\nu}_l} = 0$) decay products. On the other hand, if the decay process (4) occurs in the rest frame of the lepton l, then E_+ is the maximum total energy to the lepton l_1 while E_0 is the maximum total energies of the neutrino decay products.

Motivated by the heuristic model we develop a geometric theory by elevating the coupling constant τ to a classical parameter alongside the classical parameters mass m and charge q. Classical particles are thus triples (τ, m, q) in $\mathbb{R}^3$ that are constrained to satisfy the fundamental relation (2). The resulting $SO(1,2)$ geometry on classical particle parameter space then leads to the fundamental energy formulas (5) and (6) as the eigenstate energies of the $SO(1,2)$ metric for processes with $\Delta q = 0$. We then use a fundamental **time quantization condition** to derive the mass spectrum

$$m_0 = 84,000 \text{ MeV} \quad , \quad m_1 = 1816 \text{ MeV} \quad , \quad m_2 = 140 \text{ MeV} \quad , \quad m_3 = 105.75 \text{ MeV}$$

assuming the masses $m_0 = 84,000$ MeV and $m_e = 0.511$ MeV.

The starting point of the present work is to recall that in Dirac's 1938 derivation of the Lorentz-Dirac equation (1) fundamental use was made of the "instantaneous rest frames" of the charged particle. Since the trajectory is in general accelerated, one knows that the instantaneous rest frames along the trajectory of the charged particle are related to one another by Fermi-Walker transport. With this in mind we consider a charged particle following an accelerated trajectory $\sigma(s)$ in spacetime. Fixing a point along the trajectory, one can express the components of the acceleration vector relative to a Fermi-Walker transported tetrad, which we denote by $a^{(i)}(s)$. One finds that the first order Taylor series expansion of these components $a^{(i)}$ is

$$a^{(i)}(s) = a^{(i)}(0) + s \cdot \left(\dot{a}^{(i)} - \frac{a^2}{c^2} u^{(i)} \right)(0) \tag{7}$$

The first order term is proportional to the Abraham 4-vector

$$\frac{2e^2}{3c^3}(\dot{a}^i - \frac{a^2}{c^2} u^i) \tag{8}$$

that occurs on the right-hand-side of the Lorentz-Dirac equation (1). We use this fact to show below that the Abraham 4-vector can thus be removed by a "quantization of time" ansatz, which amounts to viewing the dynamics as being discretized into units of time defined by

$$\tau_e = \frac{2e^2}{3m_e c^3} \approx 0.6 \times 10^{-23} \text{sec} \quad .$$

Based on this electron time scale we show that the motion of a free electron is well-approximated by geodesic motion. Several important consequences follow from this "quantization of time" ansatz, namely:

a. The self-accelerating ("runaway") solutions for the electron are eliminated;

b. The self-accelerating solutions for particles with masses greater than the electron are shown to be related to particle decay; and

c. The massless neutrinos that pair off with the leptons occur naturally in the theory.

The structure of the paper is as follows. In section 2 we present our analysis of the kinematics of accelerated motion that leads to equation (7) discussed above. We then apply the result to the motion of the electron in section 3 where we show that a quantization of time ansatz eliminates the unphysical run-away solutions. Applying the result to the higher mass particles in section 4, we show that based on the electron time scale τ_e the run-away solutions can be reinterpreted in terms of particle decay processes with $\Delta q = 0$. Sections 5 and 6 are devoted to the development of the $SO(1, 2)$ geometric model for the space of classical parameters. Then in section 7 we present the heuristic model for the dynamical mass spectrum for particles that can decay to the electron. In section 8 we discuss the relationship of the theory to the standard electroweak model, and in section 9 we present a summary of the paper together with some concluding remarks based on the results of the paper.

2 Fermi-Walker frames

Let $\sigma(s)$ be an accelerating timelike curve in Minkowski spacetime, and let (x^i) denote an arbitary background inertial coordinate system. At some initial point $\sigma(s_0)$ on σ we select an arbitrary orthonormal frame and then construct an orthonormal tetrad $(e_{(k)}(s))$ along the curve by Fermi-Walker transport of the initial tetrad.

Now let V be a vector field over the curve σ. Along the curve we may express this vector field in components with respect to the background chart and also with respect to the Fermi-Walker transported frame. Thus

$$V = V^i(s)\partial_i|_{\sigma(s)} = V^{(i)}(s)e_{(i)}(s)$$

where the notational convention is that indices will be enclosed in parentheses if a tensor is referred to a Fermi-Walker frame.

Consider next a Taylor series expansion of the components $V^{(i)}(s)$ of the vector field V with respect to the FW-transported frame:

$$V^{(i)}(s) = V^{(i)}(0) + s \cdot \frac{dV^{(i)}}{ds}(0) + \dots \tag{9}$$

The goal here is to re-express this formula in terms of the FW-covariant derivatives. We do this by first differentiating the formula $V^{(i)}(s) = e^{(i)}_j(s)V^j(s)$ to obtain

$$\frac{dV^{(i)}}{ds}(s) = \frac{de^{(i)}_j}{ds}(s) \cdot V^j(s) + e^{(i)}_j(s)\frac{dV^j}{ds}$$

Next replace the first time derivative term on the RHS with the FW-transport formula $\frac{de^{(i)}_j}{ds} = \Lambda^k_j e^{(i)}_k$ where $\Lambda^k_j = \frac{1}{c^2}(a^k u_j - a_j u^k)$ is the FW-rotation matrix. Substituting we obtain

$$\frac{dV^{(i)}}{ds}(s) = \Lambda^k_j e^{(i)}_k(s) \cdot V^j(s) + e^{(i)}_j(s)\frac{dV^j}{ds} = e^{(i)}_k(s)\left(\frac{dV^k}{ds} + \Lambda^k_j V^j(s)\right) = \left(\frac{D_{(FW)}V}{Ds}\right)^{(i)}$$

Substituting this back into (9) we have our desired formula, namely

$$V^{(i)}(s) = V^{(i)}(0) + s \cdot \left(\frac{D_{(FW)}V}{Ds}\right)^{(i)}(0) + \dots \tag{10}$$

The result is that the time derivatives in the Taylor series expansion of vector components, expressed with respect to a FW-transported frame, are the FW-covariant derivatives.

Now apply this result when we take the acceleration vector a for V. Note first that the Fermi-Walker covariant derivative of a has the form

$$\frac{D_{(FW)}a}{Ds} = \frac{da}{ds} + \frac{1}{c^2}(a[u \cdot a] - u[a \cdot a]) = \dot{a} - \frac{a^2}{c^2}u$$

Substituting this into (10) we obtain the formula

$$a^{(i)}(s) = a^{(i)}(0) + s \cdot \left(\dot{a}^{(i)} - \frac{a^2}{c^2}u^{(i)}\right)(0) + \dots \tag{11}$$

This result could be anticipated by noting that the notion of *uniform acceleration* in spacetime can be defined (Rohrlich, 1965) by the vanishing of the Abraham four-vector (8).

3 The Stable Electron

Let us now consider the above description in the context of the Lorentz-Dirac equation

$$a^i = (q/mc)F^i_j u^j + \tau_e \left(\dot{a}^i - (\frac{a_j a^j}{c^2}) u^i \right)$$

We imagine that a classical electron is moving along its world line in Minkowski spacetime according to this equation of motion, and restrict attention to the case in which the external field F^i_j is zero, so the particle is a free particle. Now divide the world line into N equal pieces of proper-time length s^*, so that for $i = 0, 1, \ldots N-1$ we have $s_{i+1} = s_i + s^*$. Select one of the marked points, say s_j, along the trajectory. For small times s to the future of s_j the acceleration in the Fermi-Walker frame is given by (see 11)

$$a^{(i)}(s_j + s) = a^{(i)}(s_j) + s \cdot \left(\dot{a}^{(i)} - \frac{a^2}{c^2} u^{(i)} \right)(s_j) + \ldots \tag{12}$$

On the otherhand, the acceleration is also specified by the Lorentz-Dirac equation without the $(q/mc)F^i_j u^j$ term:

$$a^{(i)}(s_j + s) = \tau_e \left(\dot{a}^{(i)} - (\frac{a^2}{c^2}) u^{(i)} \right)(s_j + s)$$

Expanding the right-hand-side to first order in s we obtain

$$a^{(i)}(s_j + s) = \tau_e \left(\dot{a}^{(i)} - (\frac{a^2}{c^2}) u^{(i)} \right)(s_j) + \mathcal{O}(s \cdot \tau)$$

We will shortly choose $s^* = \tau_e$ so that it will be appropriate to work to first order in the parameter $\tau_e << 1$. The first term on the right-hand-side of this last equation is first order, while the remaining terms are second order and higher since they will involve products of τ_e and powers of s greater than zero. Thus to first order in τ_e we find that the value of the acceleration for small times into the future is, according to the Lorentz-Dirac equation,

$$a^{(i)}(s_j + s) = \tau_e \left(\dot{a}^{(i)} - (\frac{a^2}{c^2}) u^{(i)} \right)(s_j) \tag{13}$$

Equating the two expressions for $a^{(i)}(s_j + s)$ from (12) and (13) yields, after rearrangment of terms,

$$a^{(i)}(s_j) = (\tau_e - s^*) \left(\dot{a}^{(i)} - (\frac{a^2}{c^2}) u^{(i)} \right)(s_j) \tag{14}$$

to first order in τ_e and s^*. Observe that if we choose time steps for the electron to be $s^* = \tau_e \approx 10^{-23}$ secs, then at each of the sample points along the world line we have

$$a^{(i)}(s_j) = 0 \tag{15}$$

Hence by quantizing time the world line is divided into small segments, and at each sample point the acceleration vector vanishes. This discretized motion well-approximates continuous motion with $a^{(i)}(s) = 0$ for all values of the proper time s. We conclude that the classical electron will thus be a freely moving classical charged particle executing geodesic motion in the absence of an external field. In particular, there are no *run-away* solutions. Since the electron is the lightest and only stable lepton, we make the assumption that

ASSUMPTION #1: Lepton time is quantized in time units based on τ_e.

We have just seen that this assumption eliminates the "run-away" solutions for the classical electron. We now show that this assumption does the same thing for the higher mass particles.

4 No run-away solutions for higher mass particles

For an electron and another classical charged particle with the same electric charge the definition (2) of the parameter τ implies the equality

$$m_e c^2 \tau_e = m c^2 \tau \tag{16}$$

Hence

$$\tau < \tau_e \tag{17}$$

when $m > m_e$. Now consider a classical charged particle (τ, m, e) such that (16) and (17) are satisfied, with $m > m_e$, and let the external electromagnetic field again be the zero field. Divide the trajectory of the particle, defined by the Lorentz-Dirace equation, into N parts using the time quantization condition $\Delta s = \tau_e$. Then as we did above for the electron (see equation (14)) we arrive at the equation

$$a^{(i)}(s_j) = (\tau - \tau_e)\left(\dot{a}^{(i)} - (\frac{a^2}{c^2})u^{(i)}\right)(s_j) \quad . \tag{18}$$

By (17) $\tau - \tau_e < 0$, so $\Delta\tau := \tau_e - \tau > 0$ and hence equation (18) may be rewritten in the form

$$a^{(i)}(s_j) = -\Delta\tau\left(\dot{a}^{(i)} - (\frac{a^2}{c^2})u^{(i)}\right)(s_j) \quad . \tag{19}$$

Thus under the time quantization condition the particle experiences an acceleration given by (19) at each sample point along its trajectory. Since the time steps are roughtly 10^{-23} seconds, we make the assumption:

ASSUMPTION #2: The formula (19) holds for each value of the proper time s

This assumption allows us to rewrite (19) as

$$a^{(i)}(s) = -\Delta\tau \left(\dot{a}^{(i)} - (\frac{a^2}{c^2})u^{(i)} \right)(s) \quad . \tag{20}$$

Note that when $\Delta\tau = 0$ then this equation reduces to (15) which again implies no self-accelerating electrons. However, when $\Delta\tau > 0$, equation (20) shows that the particle will self-accelerate. We interpret equation (20) as describing a **self-interaction** phenomena, and we will show that it can in fact describe the decay of a massive particle into a pair of particles, one massive and the other massless.

Consider a process in which the particle (τ, m, e) self-interacts from rest at $s = 0$ according to (20) with $a(0) \neq 0$. We make the following assumption:

ASSUMPTION #3: The initial acceleration $a(0)$ is defined by the natural constants of the theory, namely the speed of light c and the natural time unit τ_e.

Hence $a(0)$ is given by

$$a(0) = \frac{c}{\tau_e} \quad . \tag{21}$$

Although this is an enormous number ($\approx 10^{34}$ cm/sec^2 for the electron) it may well be appropriate for a particle decay process.

The problem now is to solve (20) subject to the initial conditions $v(0) = 0$ and $a(0) = \frac{c}{\tau_e}$, where v denotes the magnitude of the proper 3-velocity. We do this by following Dirac's procedure (Dirac, 1938), the only changes being in the initial conditions and the **sign** on the right-hand side of (20). The result is

$$v(s) = c \sinh \left(\frac{\Delta\tau}{\tau_e}(1 - e^{-s/\Delta\tau}) \right) \tag{22}$$

Taking the limit as $s \longrightarrow \infty$ we determine the limiting velocity of the self-interacting particle to be

$$v^* = \lim_{s\to\infty} v(s) = c \sinh \left(\frac{\Delta\tau}{\tau_e} \right) \quad . \tag{23}$$

Observe that because of the extremely small value of $\Delta\tau$, the velocity $v(s)$ will very quickly approach the limiting value given in (23) after only a few multiples of the basic time unit τ_e. By using (16) we have the following alternative form for v^*:

$$v^* = c \sinh(\frac{\Delta m}{m}) \quad . \tag{24}$$

We now consider three simple model processes based on this result. First recall the following facts about particle decay mentioned in the introduction. Consider the relativistic kinematics of the decay process

$$P \longrightarrow P_1 + P_0 \quad , \tag{25}$$

where particle P has rest mass $m > m_1$, particle P_1 has rest mass $m_1 > 0$, and particle P_0 has rest mass $m_0 = 0$. If the decay process (25) occurs with P at rest, then from conservation of energy-momentum one knows that the total energies of particles P_1 and P_0 are given by (5) and (6), respectively.

Model I.

As discussed above we suppose first that a particle (τ, m, e) self-interacts beginning at $s = 0$ and after a long time attains the limiting velocity $v^* = c\ \sinh(1 - \frac{m_e}{m})$. We appeal to the principle of conservation of energy and suppose that total energy is conserved, and that the price that the particle pays for the self-interaction is a loss of rest mass energy. That is to say, we suppose that as the particle self-interacts its rest mass becomes a function $m(s)$ of proper time, but that total energy is conserved. Thus we assume the energy balance equation

$$m(s)c^2\gamma(s) = mc^2 \quad , \tag{26}$$

where $\gamma(s) = (1 - \frac{\mathrm{v}^2(s)}{c^2})^{-1/2}$ is the relativistic correction factor, and v denotes the three velocity measured in the Lorentz frame in which the particle is initially at rest. From (22) and (26) (with $a(0) = c/\tau_e$) we obtain

$$m(s) = m\gamma^{-1}(s) = m\,\mathrm{sech}\left(\frac{\Delta\tau}{\tau_e}(1 - e^{-s/\Delta\tau})\right) \quad . \tag{27}$$

From this equation we now find the mass decay formula

$$m(0) = m \stackrel{s\to\infty}{\longrightarrow} m(\infty) = m\,\mathrm{sech}(\frac{\Delta\tau}{\tau_e}) = m\,\mathrm{sech}(\frac{\Delta m}{m}) \quad . \tag{28}$$

Thus under the hypothesis of conservation of energy as formulated in (26) the decaying particle starts with rest mass m, and after self-interacting for a long time ends with the rest mass $m(\infty)$ given in (28).

We next expand (28) to first order to obtain

$$m(\infty) = m(1 - \frac{1}{2}(\frac{\Delta m}{m})^2) \quad . \tag{29}$$

Using $\Delta m = m - m_1$ in this last equation yields

$$m(\infty) = m_1 + \left(\frac{m^2 - m_1^2}{2m}\right) = m_1 + \frac{E_0}{c^2} \tag{30}$$

where we have used the definition (6). Thus the decaying particle experiences the mass decay

$$m \longrightarrow m(\infty) = m_1 + \frac{E_0}{c^2} \quad . \tag{31}$$

In this model the decaying particle P in (25) starts with rest mass m and ends up with rest mass $m(\infty)$ equal to the mass m_1 of particle P_1 plus the mass equivalent of the total energy of the massless particle P_0. We thus have a dynamical model for the mass transformation in a decay process of the type (25).

From (28) we can now compute the change in rest mass energy:

$$\Delta E_{RM} = mc^2 - m(\infty)c^2 = E_+ - m_1c^2 \tag{32}$$

The result is that the rest mass energy that is lost makes up the kinetic energy of the massive decay product P_1. Now rewrite equation (32) to express the energy balance equation as

$$mc^2 = m(\infty)c^2 + \Delta E_{RM} \quad . \tag{33}$$

Using (5), (6), (30) and (32) in this equation we find

$$\begin{aligned} mc^2 &= (m_1c^2 + E_0) + (E_+ - m_1c^2) \\ &= E_+ + E_0 \end{aligned} \tag{34}$$

which is the energy balance equation that one obtains from relativistic kinematics and conservation of energy-momentum for a process of the type (25).

Model II.

The mass decay formula (27) thus leads to results in agreement with standard relativistic kinematics and conservation of energy-momentum whenever $(\Delta m) << m$ so that the expansion made in (29) is valid. Taking m_1 and m as the lighter and heavier masses, respectively, for the massive particles in a decay of the type (25), then from experimental values we have

$$(a) \quad (\frac{\Delta m}{m}) \quad \approx \quad 0.224 \quad \text{for} \quad \pi^- \longrightarrow \mu^- + \bar{\nu}_\mu \quad , \tag{35}$$

$$(b) \quad (\frac{\Delta m}{m}) \quad \approx \quad 0.996 \quad \text{for} \quad \pi^- \longrightarrow e^- + \bar{\nu}_e \quad . \tag{36}$$

Thus the approximation (29) is reasonable for the decay of a pion to a muon plus antineutrino, but it is a poor approximation for the decay of a pion to an electron plus antineutrino. On the other hand there is no reason to compute the final decay mass in the limit $s \longrightarrow \infty$ since, as mentioned above, the exponential in (27) will decay toward zero very quickly. Using the approximate result (30) as a guide, we define the **decay time** $s^*(m_1, m)$ for a decay process of the type (25) by the formula

$$m^* := m(s^*(m_1, m)) = m_1 + \frac{E_0}{c^2} \quad , \tag{37}$$

where $m(s)$ is defined in equation (27). Using (27) in equation (37) we find the decay time formula $s^*(m_1, m) = -\Delta\tau(m_1, m) \ln\big(1 - \frac{\Delta m}{m} \operatorname{sech}^{-1}(\frac{m^*}{m})\big)$. With the definition (37) we now have in place of (30) the definition

$$m^* = m_1 + \frac{E_0}{c^2} \quad . \tag{38}$$

The change in rest mass energy is again given by (32), so that the energy balance equation (34) again holds. In this process a particle of rest mass m self-interacts for a time s^* given above, looses rest mass energy according to (26), and at the end of this time period ends up with rest mass energy

$$m^* c^2 = m_1 c^2 + E_0 \quad . \tag{39}$$

The change in rest mass energy given in (32) thus goes into the kinetic energy of the massive decay product.

Model III.

An alternative, and perhaps more intuitive approach is to define the decay time $s^\#$ by the formula

$$m^\# := m(s^\#(m_1, m)) = \frac{E_1}{c^2} \quad , \tag{40}$$

so that the rest mass energy that is left over after the decay period $0 \to s^\#$ is precisely equal to the total energy of the massive decay product. As in Model II we can solve (40) for $s^\#$, for which we find $s^\#(m_1, m) = -\Delta\tau(m_1, m) \ln\Big(1 - \frac{\Delta m}{m} \operatorname{sech}^{-1}(\frac{m^\#}{m})\Big)$. In this model, with conservation of total energy given as in (26), one finds that the change in rest mass energy is

$$\Delta E_{RM} = mc^2 - m^\# c^2 = E_0 \quad . \tag{41}$$

Thus the change in the rest mass energy goes over completely to the total energy of the massless decay product.

It is rather surprising that this classical self-interaction mechanism leads to a decay process that involves a massless particle. From experimental data for processes with $\Delta q = 0$ it seems clear that this massless particle cannot be the photon. We therefore identify these massless particles with the neutrinos that accompany the leptons in weak interaction decay processes. To gain further insight into this aspect of the formalism we turn next to a geometrical study of the space defined by the parameters $\tau,\ m,\ q$ for classical charged particles.

5 A geometry for classical particle space.

The results of the last few sections suggest that the classical coupling constant τ may be of fundamental importance, particularly in any process in which $\Delta q = 0$. For this reason we propose to elevate τ to the status of a classical parameter alongside the classical parameters m and q. We will implement this idea by assigning to each classical particle a triple

$$\phi = \begin{pmatrix} \tau c \\ mc \\ \frac{2q}{\sqrt{3c}} \end{pmatrix} \tag{42}$$

in the space $\mathcal{TMC} = Time \times Mass \times Charge \approx \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \subset \mathbb{R}^3$, where the components satisfy the relation

$$(\tau c)(mc) = \frac{2q^2}{3c} \quad . \tag{43}$$

Define a metric on $\mathbb{R}^3$ by

$$h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{44}$$

Then viewing each ϕ as in (42) as a vector in $\mathbb{R}^3$, the components of ϕ will satisfy the defining relation (43) provided ϕ satisfies

$$h(\phi, \phi) = 0 \quad . \tag{45}$$

The set of vectors of the form (42) satisfying this relation define a surface Σ in $\mathbb{R}^3$. Thus rather than treating the relation $\tau = \frac{2q^2}{3mc^3}$ as a **definition** of τ, we are considering the quantities τc, mc and $\frac{2q}{\sqrt{3c}}$ as **independent parameters constrained by relation (45)**. This shift in point of view allows us to use any pair of variables, in particular the pair $(\tau c, mc)$, to describe a classical particle rather than the traditional choice (m, q).

Note that each entry in the vector ϕ defined in (42) has units distinct from the other two entries. It will be convenient to initially work with dimensionless variables, and to do this we define coordinates on the space $\mathcal{TMC}$ of the triples ϕ as follows. Choose a reference particle

$$\phi_1 = \begin{pmatrix} \tau_1 c \\ m_1 c \\ \frac{2q_1}{\sqrt{3c}} \end{pmatrix} \tag{46}$$

with all entries **non-zero**. Then define a coordinate map $X_1 : \mathcal{TMC} \longrightarrow \mathbb{R}^3$ by

$$X_1(\phi) = G\phi = \begin{pmatrix} 1/\tau_1 c & 0 & 0 \\ 0 & 1/m_1 c & 0 \\ 0 & 0 & \sqrt{3c}/2q_1 \end{pmatrix} \begin{pmatrix} \tau c \\ mc \\ \frac{2q}{\sqrt{3c}} \end{pmatrix} = \begin{pmatrix} \tau/\tau_1 \\ m/m_1 \\ q/q_1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{47}$$

If both ϕ and ϕ_1 define classical particles, then both vectors lie on the surface Σ in $\mathbb{R}^3$ defined in (45). It follows that the coordinates x,y and z defined in (47) satisfy the equation $xy = z^2$. Simultaneously the metric h transforms as $h \to \bar{h} = G^{-1}hG^{-1} = (m_1\tau_1 c^2)g$ where g denotes the metric

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{48}$$

Since both $\bar{h}$ and g define the same null cone in $\mathbb{R}^3$, we will drop the multiplicative constant and work with the dimensionless metric g. This metric can be diagonalized by a rotation by $\pi/4$ about the z-axis, with the matrix of the rotation given by

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \tag{49}$$

The transformation of g given in (48) is then $g \to \bar{g} = R^t g R$, where

$$\bar{g} = R^t g R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{50}$$

Simultaneously the coordinates X_1 transform as $X_1 \to \bar{X}_1 = R^t \circ X_1$, given explicitly by

$$\bar{X}_1(\phi) := R^t \circ X_1(\phi) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x + y \\ y - x \\ \sqrt{2}z \end{pmatrix} \tag{51}$$

From (43) we have the relation

$$\frac{\tau}{\tau_1} = \left(\frac{m_1}{m}\right) \left(\frac{q}{q_1}\right)^2 \tag{52}$$

holding *on shell*, namely on the constraint surface Σ defined by (45). Using this in (51) we find *on shell* that the new barred coordinates can be expressed in terms of the original parameters as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{m^2 + m_1^2 (q/q_1)^2}{m m_1} \\ \frac{m^2 - m_1^2 (q/q_1)^2}{m m_1} \\ 2(q/q_1) \end{pmatrix} \tag{53}$$

Moreover, since *on shell* $g(X_1(\phi), X_1(\phi)) = \bar{g}(\bar{X}_1(\phi), \bar{X}_1(\phi)) = 0$ we have that the new barred coordinates satisfy the equation

$$\bar{x}^2 - \bar{y}^2 - \bar{z}^2 = 0 \tag{54}$$

which is a cone in $\mathbb{R}^3$.

Now set $q = q_1 =$ constant, so that we are considering only those classical charged particles that have the same non-zero charge q_1 as the reference particle ϕ_1. The intersection of the surface Σ by the $q = q_1$ plane gives the curve xy = constant on the surface Σ. In the new coordinates $\bar{X}_1$ this curve is described by the hyperbola

$$\bar{x}^2 - \bar{y}^2 = \text{constant} \tag{55}$$

In this case (53) reduces to

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{m^2+m_1^2}{mm_1} \\ \frac{m^2-m_1^2}{mm_1} \\ 2 \end{pmatrix} \tag{56}$$

To gain insight we next define **the energy representation coordinates** $\bar{\mathcal{E}}_1$ of $\bar{X}_1(\phi)$ with respect to ϕ_1 by

$$\bar{\mathcal{E}}_1(\phi) := \frac{m_1 c^2}{\sqrt{2}} \bar{X}_1(\phi) \quad . \tag{57}$$

From (56) and (57) we find

$$\bar{\mathcal{E}}_1(\phi) = \begin{pmatrix} (\frac{m^2+m_1^2}{mm_1})c^2 \\ (\frac{m^2-m_1^2}{mm_1})c^2 \\ m_1 c^2 \end{pmatrix} \tag{58}$$

These ϕ_1-energy coordinates have the following physical interpretation. Consider the pion decay (3). If the decay occurs in the rest frame of the pion, then conservation of energy-momentum leads to the well-known formulas (5) and (6) for the total energies of the decay particles. Thus if we interpret ϕ as π^- and ϕ_1 as the lepton-antineutrino pair, then by comparing (58) with (5) and (6) we find the following results:

The ϕ_1 energy coordinates of the transformed state

$$\bar{X}_1(\phi) = (\bar{X}_1(\phi)^1, \bar{X}_1(\phi)^2, \bar{X}_1(\phi)^3)$$

are:

- $\bar{\mathcal{E}}_1(\phi)^1 = E_+ =$ total energy of massive lepton l^- in the decay $\pi^- \longrightarrow l^- + \bar{\nu}_l$.
- $\bar{\mathcal{E}}_1(\phi)^2 = E_0 =$ total energy of massless lepton $\bar{\nu}_l$ in the decay $\pi^- \longrightarrow l^- + \bar{\nu}_l$.
- $\bar{\mathcal{E}}_1(\phi)^3 = m_1 c^2 =$ rest mass energy of massive lepton l^- in the decay $\pi^- \longrightarrow l^- + \bar{\nu}_l$.

If instead of (3) we consider the pure leptonic decay (4), then we recall that if the decay occurs in the rest frame of the massive lepton l, then the energy given in (5) is the **maximum** total energy of l_1, and (6) gives the **maximum** total energies of ν and $\bar{\nu}_1$. In the next section we use eigenvector-eigenvalue methods to give an invariant formulation of the results of this section.

6 Eigenstates.

Recall that the metric g defined in (48) is determined by the fundamental relation (43) between the variables τc, mc, and $\frac{2q}{\sqrt{3}c}$. Consider the eigenvectors and eigenvalues of g. The eigenvalues are

$$\lambda^+ = 1 \quad , \quad \lambda^- = -1 \quad , \quad \lambda^0 = -2 \quad , \tag{59}$$

and a corresponding set of g-orthonormal eigenvectors is

$$u^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad u^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad , \quad u^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{60}$$

Next we express an arbitrary particle ϕ in terms of the eigenvectors u^+ and u^- and u^0. From (47) with $q = q_1$ and $\frac{\tau}{\tau_1} = \frac{m_1}{m}$ we have

$$X_1(\phi) = \begin{pmatrix} m_1/m \\ m/m_1 \\ 1 \end{pmatrix} \tag{61}$$

Reexpressing this vector $X_1(\phi)$ as a linear superposition of the eigenstates given in (60) we find

$$X_1(\phi) = \left(\frac{\sqrt{2}E_+}{m_1 c^2} \right) u^+ + \left(\frac{\sqrt{2}E_0}{m_1 c^2} \right) u^- + \left(\sqrt{2} \right) u^0 \quad . \tag{62}$$

As above the definition of the energy representation coordinates of $X_1(\phi)$ with respect to ϕ_1 is $\mathcal{E}_1(\phi) := \frac{m_1 c^2}{\sqrt{2}} X_1(\phi)$, which now yields the decomposition

$$\mathcal{E}_1(\phi) = (E_+)\, u^+ + (E_0)\, u^- + \left(m_1 c^2 \right) u^0 \tag{63}$$

This is the invariant form of formula (58) above, and the physical interpretation of the decomposition is the following. A general particle with rest mass $m \geq m_1$ and charge $q = q_1$ is, by (63), a linear superpostion of the eigenstates of the metric g given in (48). We identify the state $(E_+)u^+$ with eigenvalue $\lambda^+ = 1$ with a massive lepton, and the eigenstate $(E_0)u^-$ with the corresponding antineutrino, rather than a neutrino, because of its negative eigenvalue $\lambda^- = -1$. The significance of the state $(m_1 c^2)\, u^0$ with eigenvalue $\lambda^0 = -2$ is less clear. Strictly speaking the state $(E_+)u^+$ should be identified with a left-handed massive lepton. Recall that the eigenstate u^0 is fixed since we have set $q = q_1$, so it does not participate in the decay process. Since it has rest mass energy $m_1 c^2$, which is the same as the rest mass erergy of the massive left-handed lepton, we identiy the state $(m_1 c^2)\, u^0$ with a right-handed massive lepton, and identify the g-metric eigenvalue $\lambda^0 = -2$ with the weak hyper-charge Y_R of the right-handed leptons.

7 Elementary Model Dynamics

Having identified fundamental eigenstates of leptons we now consider the dynamics of the interactions of the model. Since Lorentz SO(1,1) rotations transform the hyperbola given in (55) into itself, and since each point on the hyperbola corresponds to a decay state for $\phi \to \phi_1$, a Lorentz rotation should transform two such states, say ϕ_2 and ϕ_3, one into the other.

Consider two particles with $q = q_1$ and $m_2, m_3 \geq m_1$. Since the third component will now always be 1 we will go over to a two component formalism and write in place of (42) the column vectors

$$\phi_2 = \begin{pmatrix} \tau_2 c \\ m_2 c \end{pmatrix} \quad , \qquad \phi_3 = \begin{pmatrix} \tau_3 c \\ m_3 c \end{pmatrix} \tag{64}$$

Then from (52) with $q_1 = q_2 = q_3$

$$X_1(\phi_2) = \begin{pmatrix} m_1/m_2 \\ m_2/m_1 \end{pmatrix} \quad , \qquad X_1(\phi_3) = \begin{pmatrix} m_1/m_3 \\ m_3/m_1 \end{pmatrix} \tag{65}$$

Since $m_2, m_3 \geq m_1$ these coordinates may be reexpressed as

$$X_1(\phi_2) = \begin{pmatrix} e^{-\theta_2} \\ e^{\theta_2} \end{pmatrix} \quad , \qquad X_1(\phi_3) = \begin{pmatrix} e^{-\theta_3} \\ e^{\theta_3} \end{pmatrix} \tag{66}$$

where we have made the definition

$$\theta_k := \ln(\frac{m_k}{m_1}) \ \ , \ \theta_k \geq 0 \ \ . \tag{67}$$

Transforming these state vectors using the appropriate 2×2 rotation submatrix of the rotation matrix R given in (49) we obtain the new coordinate representations $\bar{X}_1(\phi_k) = R^t \cdot X_1(\phi_k)$ of the particles:

$$\bar{X}_1(\phi_2) = \sqrt{2} \begin{pmatrix} \cosh\theta_2 \\ \sinh\theta_2 \end{pmatrix} \quad , \qquad \bar{X}_1(\phi_3) = \sqrt{2} \begin{pmatrix} \cosh\theta_3 \\ \sinh\theta_3 \end{pmatrix} \tag{68}$$

One can show that, for example, $\cosh\theta_2 = \left(\frac{(m_2)^2+(m_1)^2}{2m_1 m_2}\right)$, so the coordinatization given here is simply another form of the coordinatization given in (56). It is now easy to show that the state $\bar{X}_1(\phi_2)$ is transformed into the state $\bar{X}_1(\phi_3)$ by a Lorentz rotation (boost) of the form

$$B = \begin{pmatrix} \cosh\beta_{2\to3} & \sinh\beta_{2\to3} \\ \sinh\beta_{2\to3} & \cosh\beta_{2\to3} \end{pmatrix} \tag{69}$$

where the hyperbolic angle of the transformation is given by

$$\beta_{2\to3} = \ln(\frac{m_3}{m_2}) \ \ . \tag{70}$$

Note in particular that the hyperbolic angle needed to transform particle 2 with mass m_2 into particle 3 with mass m_3 is **independent of the coordinates** X_1 because m_1 does not appear in $\beta_{2\to3}$.

7.1 A Heuristic Model for the Mass Spectrum

We again restrict attention to particles that all have the same value of electric charge as the electron, and consider such a classical particle moving along a world line in spacetime. Then using the results of the last section we associate with the particle a time dependent 2-component column vector $\Psi(t)$ and suppose that it is **parallel transported** with respect to a general SO(1,1) connection. Then $\Psi(t)$ obeys the equation

$$D_t\Psi = \frac{d\Psi}{dt} + A(t)M\Psi = 0 \tag{71}$$

where M denotes the appropriate 2×2 matrix generator of the $SO(1,1)$ group. In the unrotated coordinates (47) with the metric given by (48), the canonical form of the generator M is

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{72}$$

and this matrix generates the conjugate group $\overline{SO}(1,1) = R^{-1}SO(1,1)R$, where R is the appropriate 2×2 submatrix of the rotation matrix given in (49) above. On the other hand, in rotated coordinates (51) with the metric given by (50), the canonical form of the generator M is

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{73}$$

and this M generates the group $SO(1,1)$. In this latter case Ψ is given by (see (68) above)

$$\Psi = \sqrt{2}\begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix} \quad , \quad \theta := \ln(\frac{m(t)}{m_1}) \tag{74}$$

and when this is substituted into (71) we obtain the equation

$$\frac{d\theta}{dt} = -A(t) \quad . \tag{75}$$

Using the definition $\theta = \ln(\frac{m(t)}{m_1})$ we find for $m(t)$ the differential equation

$$\frac{dm(t)}{dt} = -mA(t) \quad . \tag{76}$$

Making the change of variable $m \to m+m_*$ on the left hand side of this equation and integrating we obtain

$$\int d(\ln(m(t) - m_*) = k - \int_0^t A(s)ds \tag{77}$$

which can be written as

$$m(t) = m_* + (m(0) - m_*)e^{-\int_0^t A(s)ds} \quad . \tag{78}$$

In order to apply this mass decay formula we note from (71) that $A(t)$ plays the role of an effective gauge potential and thus should be of the form

$$A(t) = \gamma A_\mu \frac{dx^\mu}{dt} \quad , \tag{79}$$

where A_μ is a gauge potential 1-form, $\frac{dx^\mu}{dt}$ is a characteristic 4-velocity, and γ is a coupling constant. If we make the assumption that the gauge boson mediates the decay process, and that the boson is essentially a free particle during the time of the process, then $A(t)$ may be taken to be essentially constant. We implement this assumption by writing

$$\int_0^t A(s)ds = \omega t \quad , \tag{80}$$

where

$$\omega = \frac{m(0)c^2}{\hbar} \quad . \tag{81}$$

Now (89) may be rewritten as

$$m(t) = m_* + (m(0) - m_*)e^{-\omega t} \quad . \tag{82}$$

We make the following additional assumptions:

$$\begin{array}{lrl} \text{(A.2)} & m_e & = 0.511 \text{ MeV.} \ , \quad \alpha^{-1} = 137.034 \ , \\ \text{(A.3)} & m_* & = \frac{3}{2\alpha} m_e = 105.037 \text{ MeV} \ , \\ \text{(A.4)} & t & = t_n = n\tau_* = n(\frac{2\alpha}{3})\tau_e \ , \ n = 0, 1, 2, \ldots \ , \\ \text{(A.5)} & m(0) & = 84,000 \text{ MeV} \ . \end{array} \tag{83}$$

REMARKS:

1. A.2 and A.5 assume the stated values of the rest mass energies of the electron (m_e) and the weak interaction bosons ($m(0) = m_W$).

2. Assumptions A.3 and A.4 follow from the fundamental relation

$$mc^2\tau = \frac{2q^2}{3c} \quad . \tag{84}$$

For the electron we get

$$m_e c^2 \tau_e = \frac{2\alpha}{3}\hbar \quad . \tag{85}$$

Rewriting this equation as

$$(\frac{3m_e}{2\alpha})\tau_e = \hbar \tag{86}$$

we obtain the mass

$$m_* = \frac{3m_e}{2\alpha} \tag{87}$$

that defines by (86) the **unit of action** with respect to electron time steps τ_e. Using this value back in (84) defines the corresponding unit of time τ_* to be

$$\tau_* = \frac{2\alpha}{3}\tau_e \quad . \tag{88}$$

Assumptions that are essentially equivalent to assumption (A.4) occur in a number of works related to elementary mass spectra. See, for example, the paper by Gsponer and Hurni (Gsponer and Hurni, 1996) and references therein.

3. From (92), (95) and (A.5) we obtain

$$\omega = \frac{m(0)c^2}{\hbar} = \frac{m(0)c^2\tau_{m(0)}}{\hbar}\frac{1}{\tau_{m(0)}} = (\frac{2\alpha}{3})\frac{m(0)}{m_*}\frac{1}{\tau_*} \quad .$$

Hence

$$\omega\tau_* = (\frac{2\alpha}{3})^2\frac{m(0)}{m_e} = 3.891 \quad . \tag{89}$$

Putting the results together we may now write (93) as

$$m_n = 105.037 + (83,895)e^{-3.891n} \quad \text{MeV} \quad . \tag{90}$$

Computing the first few masses from this formula we find

$$\begin{aligned} m_0 &= 84,000 \quad \text{MeV} \\ m_1 &= 1819 \quad \text{MeV} \\ m_2 &= 140.06 \quad \text{MeV} \\ m_3 &= 105.75 \quad \text{MeV} \\ m_4 &= 105.05 \quad \text{MeV} \\ &\dots \\ m_\infty &= 105.037 \quad \text{MeV} \quad . \end{aligned} \tag{91}$$

We cut off the spectrum at m_3 since there are only three known leptons. The masses m_1, m_2 and m_3 are close to the experimental values for the tauon, the (non-lepton) pion, and the muon, respectively. There are, of course, various other ways of getting at mass spectra similar to this (Gsponer and Hurni, 1996), with perhaps the most well-known being the formula found by Barut (Barut, 1979).

8 Relationship to the Standard Model

We have seen that classical particles can be represented by triples $\phi = (\tau c, mc, \frac{2q}{\sqrt{3c}})$ in $\mathbb{R}^3$ that lie on the null cone of the metric defined in (44). Moreover, when all particles have the same magnitude of charge $|q_1| = |e|$ as the electron the formalism can be reduced to the two component formalism of the last section where now the parameters satisfy the relation

$$m_l \tau_l = \frac{2e^2}{3c^3} = \frac{2\alpha\hbar}{3c^2} \tag{92}$$

Here we have labeled the parameters with the subscript l to denote *leptons*, and for simplicity we used the definition of the fine structure constant α to rewrite the basic relationship between τ and m when $q = e$.

In order to better understand the significance of the new parameter τ we compare the parameters used here with the parameters that enter into the standard electroweak theory of Glashow, Salam and Weinberg. Recall that the relevant parameters for the leptons are

- g_l = the coupling constant for lepton l,
- v = the vacuum expectation value of the Higgs field,
- m_l = the mass of lepton l,

and that these parameters are related by the fundamental formula

$$m_l = \frac{v g_l}{\sqrt{2}} \tag{93}$$

From equations (92) and (93) we find $\frac{m_{l_2}}{m_{l_1}} = \frac{\tau_{l_1}}{\tau_{l_2}} = \frac{g_{l_2}}{g_{l_1}}$ holding for all leptons l_i, $i = 1, 2, \ldots, N$, where N is the number of leptons. Hence there exists a universal constant, say k_0, such that

$$g_{l_i} = \frac{k_0}{\tau_{l_i}} \tag{94}$$

Using (92) and (93) together with $k_0 = \tau_{l_i} g_{l_i}$ it is easy to see that

$$k_0 = \frac{2\alpha\hbar}{3vc^2} \tag{95}$$

If we consider the Higgs boson as a classical particle and identify the parameter v with the mass m_H of the Higgs boson, then this last equation together with (92) shows that $k_0 = \tau_H$, and therefore

$$g_{l_i} = \frac{\tau_H}{\tau_{l_i}} \tag{96}$$

Of course in writing τ_H we are assuming that the Higgs particle is charged as it is in various generalizations of the standard model. When this is the case we have the interpretation that the dimensionless lepton coupling constants g_{l_i}, which must be inserted into the electroweak theory by hand, are defined by the ratios of the Higgs field time parameter to the lepton time parameters.

Finally we point out that the fundamental 2×2 matrices of our reduced two-component formalism, namely the metric $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the generator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, are precisely the Pauli matrices σ^1 and σ^3, respectively. Since these two matrices are two of the three generators of the $su(2)$ Lie subalgebra of the $su(2) \times u(1)$ algebra of the standard electroweak theory, the geometry of classical particle parameter space partially defines the symmetry group of the electroweak theory.

9 Conclusions

In this paper we have re-examined the Lorentz-Dirac theory of the classical electron in an effort to determine if some of the unphysical aspects associated with the radiation reaction term in the Lorentz-Dirac equation might have alternative, physical interpretations. We have shown that the "run-away" solutions of the force-free Lorentz-Dirac equation can be eliminated using a "quantization of time" ansatz. More specifically, in sections 3 and 4 we have shown that for the electron, the lightest of the known massive leptons, the time quantization ansatz leads to the "geodesic motion" solution when applied to the force-free Lorentz-Dirac equation. For the higher mass particles the time quantization ansatz, based on the electron time scale, replaces the run-away solutions with solutions that approach a finite limiting proper 3-velocity given in (24). This limiting velocity formula was then used in three very simple heuritic models to show that in the force-free setting, the former run-away solutions could be re-interpreted and related to fundamental decay processes like (3) and (4). The intriguing aspect of these heuristic models is that the formulas imply the existence of zero mass particles associated with the decay of massive particles, in agreement with standard relativistic energy-momentum conservation results.

The new element in our analysis is the time parameter τ defined in (2) above, and which roughly characterizes the unphysical aspects of the Lorentz-Dirac equation. For an electron $\tau_e \approx 10^{-23}$ secs which is of the order of the life time of the weak interaction bosons. The results of sections 3–4 suggest that the coupling constant τ should be elevated to the status of a classical parameter along side the classical parameters mass m and charge q. We implemented this idea by introducing a geometry into the space of classical parameters. Thus rather than considering τ as being defined as in (2) by m and q, we considered τ, m and q as independent parameters that must satisfy one geometrical condition in order to represent a classical particle. Specifically, a classical particle defines a triple $\phi := (\tau c, mc, 2q/\sqrt{3c})$ in $\mathbb{R}^3$. Conversely, an arbitrary triple of this form will define a classical particle if ϕ satisfies the geometrical null vector condition $h(\phi, \phi) = 0$ where h is the metric on $\mathbb{R}^3$ defined in (44) above. This geometrical condition replaces the "definition of τ" in terms of m and q, and it has the advantage that one may use any two of these parameters, and in particular the pair $(\tau c, mc)$, to describe a classical particle. Physical states for classical particles are thus the "on shell" states that lie on the null-cone of the h-metric in $\mathbb{R}^3$.

The resulting geometry on classical particle parameter space is an $SO(1, 2)$ geometry. Since we are primarily concerned in this paper with processes in which $\Delta q = 0$, we restricted attention to the subset of particles that all have the same value $q_1 \neq 0$ of the electric charge. This restricted geometry is an $SO(1,1)$ geometry based on the $\tau - m$ part of the parameter space metric. After introducing "ratio coordinates" (see equation (47)) based on a particle with non-zero values τ_1, m_1 and q_1, we showed, by rotating these coordinates by $\pi/4$, that these coordinates in fact represent the decay state energies in processes like (3) and (4). The two eigenstates of the parameter space metric corresponding to the τ–m sector where then identified with left-handed antineutrinos and their corresponding massive leptons. We also saw that the third eigenstate of the parameter space metric could be identified with right-handed leptons, with the metric eigenvalue $\lambda = -2$ equal to the weak hypercharge Y_R of such particles. We then used these facts to develop a simple dynamical model that uses an intermediate vector boson type assumption. The resulting heuristic model led to the mass spectrum (91). The predicted spectrum, based on the assumed masses of the electron and the W-bosons, includes masses that are close to the masses of the tauon, the pion, and the muon.

Finally, in the last section we showed how the present model is related to the standard electroweak model. In particular, we found that the electroweak coupling constant g_{l_i} for the i^{th} lepton can be defined in terms of the intrinsic time parameters τ_H, of a charged Higgs particle, and τ_{l_i} of the lepton by the formula $g_{l_i} = \frac{\tau_H}{\tau_{l_i}}$. Moreover we pointed out that two of the 3 Pauli matrices that make up the standard basis of $su(2)$, namely σ^1 and σ^3, occur naturally in this model as the $SO(1,1)$ metric and the standard generator of $SO(1,1)$. Whether or not the full $SO(1,2)$ theory can generate all 4 generators of the standard electroweak $su(2) \times u(1)$ remains an open question at this point. We hope to return to these and related ideas in future publications.

References

[1] Barut, A. O., Physical Review Letters, **42**, 1251 (1979).

[2] Dirac, P. A. M., Proc. Royal Soc. (London) **A167**, 148 (1938).

[3] Glashow, S. L., Rev. Mod. Phys., **52** 539 (1980)

[4] Gsponer, A. and Hurni, J., Hadronic Journal, **19**, pp. 367-373 (1996).

[5] Jackson, J. D., *Classical Electrodynamics* (Wiley and Sons, Inc., New York, London, Sydney, 1967).

[6] Rohrlich,F., *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[7] Salam, A., Rev. Mod. Phys., **52**, 525 (1980)

[8] Weinberg, S., Rev. Mod. Phys., **52**, 515 (1980)