Symplectic Geometry on T^*M derived from n-symplectic Geometry on LM [‡]

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ABSTRACT

We establish the relationship between n-symplectic geometry on the bundle of linear frames LM of an n-dimensional manifold M and canonical symplectic geometry on the cotangent bundle T^*M. We show that all basic features of the canonical symplectic geometry of polynomial observables on T^*M, including the momentum mapping associated with *Diff(M)*, are induced from the n-symplectic geometry on LM. Moreover, the $\mathbb{C}^\times$ bundle $L^\times$ over T^*M associated with geometric quantization theory is identified with a fiber bundle associated to the principal bundle of affine frames AM of the manifold M. Viewing AM as a principal $\mathbb{R}^n$ bundle over LM we show that the connection on $L^\times$ used in geometric quantization theory is induced from a canonical connection on AM that is constructed from the $\mathbb{R}^n$-valued n-symplectic potential. We then show that the connection preserving vector fields on $L^\times$ that are related to linear polynomial observables on T^*M are also induced from connection preserving vector fields on AM.

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1. Introduction

The bundle of linear frames LM of an n-dimensional manifold M plays an especially important role in the theory of the differential geometry of the manifold [1,2,3]. This is so since once the concepts of linear connection and exterior covariant differentiation with respect to a linear connection are defined on LM, one may then use these ideas to induce connections and covariant differentiation on the tensor bundles T^r_sM over the manifold. The basic unifying element is the fact that each tensor bundles T^r_sM may be considered as a fiber bundle associated to LM via the standard action of the structure group $GL(n)$ of LM on $T^r_s\mathbb{R}^n$.

With this in mind one is led to ask if it is possible to trace other geometrical structures on tensor bundles back to the bundle of linear frames LM. Consider in particular the canonical symplectic structure on the cotangent bundle T^*M that is the basic building block of Hamiltonian dynamics when M is the configuration space of a mechanical system. The canonical symplectic structure on T^*M is $d\vartheta$ where ϑ is the canonical one-form that plays the role of a globally defined symplectic potential. Since T^*M may be considered as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$ one may ask if ϑ has its roots in a more basic structure on LM. The obvious candidate for a generalized symplectic potential on LM is the $\mathbb{R}^n$-valued soldering one-form θ since the definitions of ϑ and θ are so similar. This observation led the author to investigate whether or not one may use the **vector-valued** soldering one-form θ as a generalized symplectic potential, and the basic features of the geometry on LM that one may build up based on the generalized symplectic structure $d\theta$ may be found in [4]. The generalized symplectic geometry based on the pair $(LM, d\theta)$ will be referred to as **n-symplectic geometry**. The fact that $d\theta$ is $\mathbb{R}^n$-valued rather than $\mathbb{R}$-valued introduces new and interesting features into the geometry.

More recently the exact relationship between the canonical one-form ϑ on T^*M and the soldering one-form θ on LM was provided by Sniatycki. He showed [5] that

$$\vartheta_{[(u,\alpha)]}(\tilde{X}) = < \theta_u(X), \alpha > \quad . \tag{1.1}$$

In this equation $u = (m, \underline{e})$ denotes a point in LM that corresponds to the linear frame $\underline{e} = (e_i)$ at $m \in M$, and $[(u, \alpha)]$ denotes a point (equivalence class) in T^*M thought of as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$. In addition $\tilde{X}$ is a tangent vector at $[(u, \alpha)]$ that projects to the same vector as does the tangent vector X at $(m, \underline{e})$, and the brackets denote that natural inner product of elements of $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. Thus the fundamental building block ϑ for canonical symplectic geometry on T^*M is induced from the soldering one-form θ on LM. This raises the question: To what extend is the symplectic geometry on T^*M induced from the n-symplectic geometry on LM? The purpose of this paper is to provide some answers to this question. What we will show is that the symplectic geometry of polynomial observables on $(T^*M\ , d\vartheta)$ is induced from n-symplectic geometry on $(LM, d\theta)$. In addition the symplectic action of *Diff(M)* on T^*M together with the associated momentum mapping will be shown to be induced from an n-symplectic action of *Diff(M)* on LM and an associated n-momentum mapping.

Since the subalgebra of polynomial observables on T^*M plays a distinguished role in the theory of geometric quantization [6,7] on T^*M , the above results suggest that the geometrical structures related to the $\mathbb{C}^\times$ bundle $L^\times \to T^*M$ in geometric quantization theory might also be traceable back to LM. Indeed, we will show that $L^{\times}$ may be constructed as a fiber bundle associated to the bundle of affine frames AM of M thought of as a principal $\mathbb{R}^n$ bundle over LM. Moreover, the basic connection on $L^{\times}$ used in geometric quantization theory will be shown to be induced from a connection on $AM \to LM$ constructed from the n-symplectic potential. The results presented here will thus provide a foundation for a geometric quantization theory based on n-symplectic geometry.

The structure of the paper is as follows. In Section 2 we provide a survey of the basics of n-symplectic geometry on the frame bundle LM of an n-dimensional manifold M, and the symmetric $T^p\mathbb{R}^n$-valued observables on LM are shown to induce the homogeneous polynomial observables on T^*M . Then in Section 3 we develop the concept of momentum mappings in n-symplectic geometry. The natural action on LM of the the group *Diff(M)* of diffeomorphisms of the base manifold M is an n-symplectic action in the sense that it leaves $d\theta$ invariant. This action of *Diff(M)* on LM is then shown to induce the standard symplectic action of *Diff(M)* on $(T^*M, d\vartheta)$, and the associated n-symplectic momentum mapping is shown to induce the standard momentum mapping on T^*M associated with *Diff(M)*.

In Section 4 we consider a basic problem for a geometric quantization theory based on $(LM, d\theta)$. We use the n-symplectic potential θ to construct a canonical connection σ on the $\mathbb{R}^n$ principal bundle $AM \to LM$, and use the connection to lift the Hamiltonian vector fields on LM of rank 1 observables associated with the n-momentum mapping determined by *Diff(M)*. We thereby obtain an isomorphism of the Lie algebra of rank 1 observables on LM with a Lie algebra of connection preserving vector fields on AM. This isomorphism provides the correct Dirac canonical quantization rules for the n-symplectic momentum and position type variables.

In Section 5 we define a left action of the affine group $A(n) = GL(n) \ltimes \mathbb{R}^n$ on $\mathbb{R}^{n*} \times \mathbb{C}^{\times}$ and then show that the associated fiber bundle to $AM \to M$ determined by this action may be identified with the trivial $\mathbb{C}^{\times}$ bundle $\pi : L^{\times} \to T^*M$ used in geometric quantization theory. Using a standard technique we then use the connection σ on $AM \to LM$ to induce a connection $\tilde{\sigma}$ on $L^{\times}$. This induced connection is the connection $\pi^*(\vartheta) + \frac{1}{2\pi i}\frac{dz}{z}$ used in geometric quantization theory on T^*M . We show that the vector fields on $L^{\times}$ that are used to construct the quantum operators for the linear polynomial observables on T^*M are induced from corresponding vector fields on $AM \to LM$. We also show in Section 5 that the Hamiltonian vector fields of symmetric tensorial observables on LM map to Hamiltonian vector fields on T^*M. Section 6 consists of a set of examples of specific n-symplectic momentum mappings, and in Section 7 we present conclusions and a discussion of future work.

2. Survey of n-symplectic Geometry on LM

The principal fiber bundle $\pi_{LM} : LM \longrightarrow M$ of linear frames of an n-dimensional manifold M is the set of pairs (m, e_i) where (e_i) , $i = 1, 2, \ldots, n$ is a linear frame at $m \in M$. The dimension of LM is the even number $n(n+1)$, and the general linear group $GL(n)$ acts on LM on the right by

$$(m, e_i) \cdot g = (m, e_i g^i_j) \tag{2.1}$$

for each $g = (g^i_j) \in GL(n)$. Let (x^i) be a coordinate chart on $U \subset M$. Define canonical coordinates (x^i, π^j_k) on $\hat{U} = \pi^{-1}_{LM}(U) \subset LM$ by

$$\begin{aligned} x^i(m, e_i) &= x^i(m) \\ \pi^j_k(m, e_i) &= e^j\left(\frac{\partial}{\partial x^k}\right) \end{aligned} \tag{2.2}$$

where (e^i) denotes the coframe dual to (e_i). Moreover in (2.2) we follow standard conventions and write x^i in place of $x^i \circ \pi_{LM}$.

Let (r_i), $i = 1, 2, \ldots, n$, denote the standard basis of $\mathbb{R}^n$. Then the $\mathbb{R}^n$-valued **soldering one-form** $\theta = \theta^i r_i$ on LM may be defined by

$$\theta(X_u) = u^{-1}(d\pi_{LM}(X_u)) \quad , \quad X_u \in T_u LM \tag{2.3}$$

where $u = (m, e_i) \in LM$ is viewed as the non-singular linear map $u : \mathbb{R}^n \to T_{\pi_{LM}(u)} M$ given by $u(\xi^i r_i) = \xi^i e_i$.

The theory of n-symplectic geometry on LM developed in [4] is based on generalizing the basic structure equation

$$df = -X_f \;\lrcorner\; d\vartheta \tag{2.4}$$

on T^*M to $(LM, d\theta)$. In (2.4) f denotes any smooth $\mathbb{R}$-valued function on T^*M . Since $d\theta$ is $\mathbb{R}^n$-valued the range of the variables changes in n-symplectic geometry. The simpliest generalization of (2.4) is

$$d\hat{f} = -X_{\hat{f}} \;\lrcorner\; d\theta \tag{2.5}$$

where now $\hat{f}$ is a smooth $\mathbb{R}^n$-valued function on LM. We note that $d\theta$ is non-degenerate in the sense that

$$X \;\lrcorner\; d\theta = 0 \Longleftrightarrow X = 0 \quad . \tag{2.6}$$

Hence if a vector field $X_{\hat{f}}$ satisfies (2.5) for a given $\mathbb{R}^n$-valued function $\hat{f}$ then it will be unique. On the other hand the soldering one-form θ transformations tensorially under right translations R_g for $g \in GL(n)$ according to $R^*_g \theta = g^{-1} \cdot \theta$. A consequence of this tensorial nature of θ is that not every $\mathbb{R}^n$-valued function on LM is compatible with equation (2.5). On the other hand all smooth $\mathbb{R}$-valued functions on T^*M are compatible with equation (2.4).

Let T^1 denote the set of $\mathbb{R}^n$-valued functions $\hat{f}$ on LM that transform tensorially under right translation by $R^*_g \hat{f} = g^{-1} \cdot \hat{f}$. Such functions are in one-one correspondence with vector fields on M. Denote by HF^1 the set of $\mathbb{R}^n$-valued functions on LM that are compatible with (2.5). In [4] it is shown that

$$HF^1 = T^1 \oplus C^\infty(M, \mathbb{R}^n) \tag{2.7}$$

where the second factor denotes the smooth $\mathbb{R}^n$-valued functions on LM that are invariant on fibers. For each $\hat{f} \in HF^1$ equation (2.5) assigns a unique Hamiltonian vector field $X_{\hat{f}}$. The **Poisson bracket** of $\hat{f}, \hat{g} \in HF^1$ is defined by

$$\{\hat{f}, \hat{g}\} = X_{\hat{f}}(\hat{g}) \quad , \tag{2.8}$$

and HF^1 is a Lie algebra under this bracket. Denote by HV^1 the set of Hamiltonian vector fields $X_{\hat{f}}$ determined by elements of HF^1. Then one shows that

$$[X_{\hat{f}}, X_{\hat{g}}] = X_{\{\hat{f},\hat{g}\}} \tag{2.9}$$

so that HV^1 forms a Lie algebra.

From (2.5) it is clear that the constant $\mathbb{R}^n$-valued functions in $C^\infty(M, \mathbb{R}^n) \subset HF^1$ are all mapped to the zero vector field. Identifying these constant functions with $\mathbb{R}^n$ we have that as Lie algebras

$$HV^1 = HF^1/\mathbb{R}^n \quad . \tag{2.10}$$

Strictly speaking the bracket defined in (2.8) is not a Poisson bracket but simply a Lie bracket. However the bracket becomes a true Poisson bracket when HF^1 is combined with the higher rank observables. We denote the vector space of symmetric $\otimes^p_s R^n$-valued tensorial functions on LM by $ST^p = \{\hat{f} : LM \to \otimes^p_s R^n | \hat{f}(u \cdot h) = h^{-1} \cdot \hat{f}(u) \quad \forall \ h \in GL(n)\}$, where $\otimes_s$ denotes the symmetric tensor product, and denote the vector space of symmetric rank p contravariant tensor fields on M by $\mathrm{S}\mathcal{X}^p$. An element of ST^p corresponds to a unique element of $\mathrm{S}\mathcal{X}^p$. We denote by $ST = \sum_{p=1}^\infty ST^p$ the infinite dimensional vector space which is the direct sum of the vector spaces ST^p.

An element $\hat{f} \in ST^p$ determines [4] an equivalence classes $[\![X_{\hat{f}}]\!]$ of $\binom{n+p-2}{p-1}$ vector fields $[\![X_{\hat{f}}]\!]^{i_1\cdots i_{p-1}}$ via the n-symplectic structure equation

$$d\hat{f}^{i_1\cdots i_p} = -p! X_{\hat{f}}^{(i_1\cdots i_{p-1}} \;\lrcorner\; d\theta^{i_p)} \tag{2.11}$$

where round brackets on indices denotes symmetrization. We note that although $d\theta$ is nondegenerate in the sense of (2.6), because of the symmetrization in (2.11) the non-degeneracy is lost. For a given $\hat{f} \in ST^p$ equation (2.11) only determines the vector fields $X_{\hat{f}}^{i_1\cdots i_{p-1}}$ up to addition of vector fields $Y^{i_1\cdots i_{p-1}}$ satisfying the kernel equation

$$Y^{(i_1\cdots i_{p-1}} \;\lrcorner\; d\theta^{i_p)} = 0 \quad . \tag{2.12}$$

If a set of vector fields $Y^{i_1\cdots i_{p-1}}$ satisfies (2.12) then each vector field $Y^{i_1\cdots i_{p-1}}$ must be vertical. For a given $\hat{f} \in ST^p$ equation (2.11) thus determines an **equivalence class of $\otimes^{p-1}_s \mathbb{R}^n$-valued Hamiltonian vector fields** ($[\![X_{\hat{f}}]\!]^{i_1\cdots i_{p-1}}$), where two $\otimes^{p-1}_s \mathbb{R}^n$-valued vector fields are equivalent if their difference satisfies equation (2.12).

An element $\hat{f} = \hat{f}^{i_1 i_2\cdots i_p} r_{i_1} \otimes_s r_{i_2} \cdots \otimes_s r_{i_p} \in ST^p$ has the local canonical coordinate representation

$$\hat{f}^{i_1 i_2\cdots i_p} = f^{j_1 j_2\cdots j_p}(x) \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_p}_{j_p} \quad . \tag{2.13}$$

The associated equivalence classes of Hamiltonian vector fields $[\![X_{\hat{f}}]\!]^{i_1 i_2\cdots i_{p-1}}$ determined by equation (2.11) have the local coordinate representations [4]

$$\begin{aligned} X_{\hat{f}}{}^{i_1 i_2\cdots i_{p-1}} &= \frac{1}{(p-1)!} f^{j_1 j_2\cdots j_{p-1} k}(x) \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &\quad - \frac{1}{p!} \left(\frac{\partial f^{j_1 j_2\cdots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T_a^{i_1 i_2\cdots i_{p-1} b} \right) \frac{\partial}{\partial \pi^b_a} \end{aligned} \tag{2.14}$$

where the components $T_a^{i_1 i_2 \ldots i_{p-1} b}$ must satisfy

$$T_a^{(i_1 i_2 \ldots i_{p-1} b)} = 0 \tag{2.15}$$

but are otherwise arbitrary.

The fact that one obtains equivalence classes of vector fields rather than vector fields for the higher rank observables does not interfer with the basic algebraic structures in n-symplectic geometry. For each $p \geq 1$ the set of equivalence classes of $\otimes_s^{p-1}\mathbb{R}^n$-valued vector fields on LM, with equivalence defined as above, forms an infinite dimensional vector space. Denote by $HV(ST^p)$ the vector subspace of $\otimes_s^{p-1}\mathbb{R}^n$-valued equivalence classes of vector fields determined by elements of ST^p by equation (2.11). For $\hat{f} \in ST^p$ and $\hat{g} \in ST^q$ define the Poisson bracket $\{\ ,\ \} : ST^p \times ST^q \to ST^{p+q-1}$ by

$$\{\hat{f}, \hat{g}\}^{i_1 i_2 \ldots i_{p+q-1}} = p! X_{\hat{f}}{}^{(i_1 i_2 \ldots i_{p-1}} \left(\hat{g}^{i_p i_{p+1} \ldots i_{p+q-1})} \right) \tag{2.16}$$

where $X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}$ is any representative of the equivalence class $[\![X_{\hat{f}}]\!]^{i_1 i_2 \ldots i_{p-1}}$. The bracket so defined is easily shown to be independent of the choice of representatives and has all the properties of a Poisson bracket. In particular the bracket acts as a derivation on the commutative algebra $(ST, \otimes_s)$. Moreover, when the bracket defined here is reexpressed on the base manifold M, it gives [4] the differential concomitant of Schouten and Nijenhuis [8,9] of the symmetric tensor fields corresponding to $\hat{f}$ and $\hat{g}$. In summary we have:

Theorem 2.1 The space ST of symmetric tensorial functions on LM is a Poisson algebra with respect to the Poisson bracket defined in (2.16).

It is convenient to introduce the multi-index notation $r_{i_1 i_2 \ldots i_{p-k}} \equiv r_{i_1} \otimes_s r_{i_2} \cdots \otimes_s r_{i_{p-k}}$ for $0 \leq k \leq p-1$. Let $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}]\!]^{i_1 i_2 \ldots i_{p-1}} r_{i_1 i_2 \ldots i_{p-1}}$ and $[\![\hat{X}_{\hat{g}}]\!] = [\![X_{\hat{g}}]\!]^{i_1 i_2 \ldots i_{q-1}} r_{i_1 i_2 \ldots i_{q-1}}$ denote the vector valued equivalence classes of vector fields determined by $\hat{f} \in ST^p$ and $\hat{g} \in ST^q$. Define a bracket by

$$\begin{aligned} [[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] &= [[\![X_{\hat{f}}]\!]^{i_1 i_2 \ldots i_{p-1}}\ ,\ [\![X_{\hat{g}}]\!]^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}} \\ &= [X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}\ ,\ X_{\hat{g}}{}^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}} \end{aligned} \tag{2.17}$$

where the bracket on the right-hand-side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives. One shows that

$$[X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}\ ,\ X_{\hat{g}}{}^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}} \in [\![X_{\{\hat{f}, \hat{g}\}}]\!]\ , \tag{2.18}$$

and thus the bracket defined in (2.17) is well-defined, and we write

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = [\![\hat{X}_{\{\hat{f}, \hat{g}\}}]\!]\ . \tag{2.19}$$

Moreover, the bracket defined in (2.17) is anti-symmetric. Denote the direct sum of the vector spaces $HV(ST^p)$ by $HV(ST)$.

Theorem 2.2 The vector space $HV(ST)$ of vector valued equivalence classes of Hamiltonian vector fields on LM is a Lie algebra with respect to the bracket defined in (2.17).

Formula (1.1) above shows that the canonical one-form on T^*M is induced from the soldering one-form on LM. We show here that the polynomial observables on T^*M are induced from related objects on LM. In particular elements of ST^p induce degree p homogeneous polynomial observable on T^*M as follows. Consider T^*M as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$. Then for $\hat{f} \in ST^p$ define $\tilde{f} : T^*M \to \mathbb{R}$ by

$$\tilde{f}([u, \alpha_i]) = < \hat{f}(u), \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p} > \tag{2.20}$$

where $[u, \alpha_i] \in T^*M$, $u = (m, e_j) \in LM$, and the brackets denote the extended natural inner product of elements of $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. The tensorial character of $\hat{f}$ guarantees that this definition is independent of choice of representatives of the equivalence class $[u, \alpha]$.

From (2.2) we note that $\pi^i_j(m, e_k)\alpha_i = e^i(\frac{\partial}{\partial x^j})\alpha_i = p_j(e^i\alpha_i)$ where (p_j) are the standard momentum coordinates on T^*M defined by the local chart (x^i) on M. Then, for example, for p=2 take $\hat{f} = \hat{f}^{ij} r_i \otimes_s r_j$ where $\hat{f}^{ij} = f^{ab}(x)\pi^i_a\pi^j_b$. The definition (2.20) yields

$$\tilde{f}([m, e_j, \alpha_i] = f^{ab}(x)p_a p_b \quad , \tag{2.21}$$

which is a homogeneous quadratic polynomial observable on T^*M . At the end of Section 5 we show that the equivalence class of Hamiltonian vector fields $[\![X_{\hat{f}}]\!]$ for $\hat{f} \in ST^p$ may be mapped to the Hamiltonian vector field $X_{\tilde{f}}$ of $\tilde{f}$ on T^*M , where $\tilde{f}$ is induced from $\hat{f}$ as in (2.20).

In general, it can be observed that n-symplectic geometry selects "allowable observables" in the sense that not every $\otimes^p_s R^n$-valued function on LM is compatible with (2.11). It is known [4] that the most general $\otimes^p_s R^n$-valued function on LM that can satisfy (2.11) for some set of vector fields must be a polynomial in the momentum coordinates with coefficients in the set of functions that are invariant on fibers on LM. We denote this set by SHF^p. For a given $p \geq 1$ the homogeneous degree p polynomials in SHF^p form the set ST^p, while for $p > 2$ the lower degree polynomials do not in general correspond to elements of ST^q for $0 \leq q < p$. The reader is referred to [4] for more details.

3. n-symplectic Momentum Mappings

In the applications of symplectic geometry to classical and quantum mechanics the concept of momentum mapping [10] plays an especially important role. In classical mechanics it provides a geometrization of conservation laws associated with Hamiltonian systems with symmetries, and in geometric quantization the momentum mapping on T^*M associated with the action of *Diff(M)* is fundamentally related to the geometrization of the Dirac canonical quantization procedure. Here we introduce the concept of momentum mappings in n-symplectic geometry on $(LM, d\theta)$. In the following an n-symplectic action of a Lie group G on LM is an action that preserves the n-symplectic form $d\theta$.

DEF. #3.1: Let $\Phi : G \times LM \rightarrow LM$ be an n-symplectic action of a Lie group G on the n-symplectic manifold $(LM, d\theta)$. Then a mapping $J : LM \rightarrow \mathcal{G}^* \otimes \mathbb{R}^n$ is a momentum mapping if for each $\xi \in \mathcal{G}$

$$d\hat{J}(\xi) = -\xi^* \quad \lrcorner \; d\theta \tag{3.1}$$

where ξ^* is the infinitesimal generator of the action of G on LM generated by ξ, and $\hat{J}(\xi) : LM \rightarrow \mathbb{R}^n$ is defined by

$$\hat{J}(\xi)(u) = < J(u), \xi > \quad . \tag{3.2}$$

The brackets in (3.2) denote the natural inner product of elements of $\mathcal{G}^*$ and $\mathcal{G}$.

This definition generalizes the definition of momentum mapping on a general symplectic manifold, the main difference being that the range is now $\mathcal{G}^* \otimes \mathbb{R}^n$ rather than $\mathcal{G}^*$.

To obtain a specific example of an n-symplectic momentum mapping we consider the Lie group $G =$ *Diff(M)* of diffeomorphisms of the base manifold M. The Lie algebra $\mathcal{G}$ of *Diff(M)* is the set of smooth vector fields on M. Let $\Phi : G \times M \rightarrow M$ denote the group action. This action lifts to a left action of G on LM in a natural way [1]. For each $f \in G$ the associated map $\Phi_f : M \rightarrow M$ induces a mapping $\tilde{\Phi}_f : LM \rightarrow LM$ defined by

$$\tilde{\Phi}_f(m, e_i) = (\Phi_f(m), \Phi_{f*}(e_i)) \quad . \tag{3.3}$$

Then the action $\tilde{\Phi} : G \times LM \rightarrow LM$ of G on LM is $\tilde{\Phi}(f, u) = \tilde{\Phi}_f(u)$ for $u \in LM$. It is known ([1], page 226) that the soldering 1-form θ is invariant under this action. Hence the action of *Diff(M)* on LM defined in (3.3) is an **n-symplectic action**.

Next consider a vector field $X \in \mathcal{G}$. X generates a local 1-parameter group φ_t of local diffeomorphisms of M which in turn lifts to a local 1-parameter group $\tilde{\varphi}_t$ of local diffeomorphisms of LM. The infinitesimal generator X^* of $\tilde{\varphi}_t$ is referred to as the **natural lift** of X [1]. It follows from $L_{X^*}\theta = 0$ that

$$d(X^* \lrcorner \; \theta) = -X^* \lrcorner \; d\theta \quad . \tag{3.4}$$

Moreover since $R_{g*}(X^*) = X^*$ it follows that $X^* \lrcorner \; \theta \in T^1 \subset HF^1$. Hence the subset T^1 of rank 1 Hamiltonian functions HF^1 on LM is uniquely related to the group *Diff(M)*.

We can now exhibit a momentum mapping associated with the action of *Diff(M)* on LM. For each vector field $X \in \mathcal{G}$ define $\hat{J}(X) : LM \rightarrow \mathbb{R}^n$ by

$$\hat{J}(X) = \hat{X} \tag{3.5}$$

where $\hat{X}$ is the $\mathbb{R}^n$-valued tensorial function in T^1 uniquely determined by X. In particular $\hat{X} = X^* \lrcorner \; \theta$. As discussed above we know that

$$d\hat{J}(X) = -X^* \quad \lrcorner \; d\theta \tag{3.6}$$

so that (3.5) satisfies the definition of an n-symplectic momentum mapping. We observe that the set of all $\hat{J}(X)$ for $X \in \mathcal{G}$ is thus the subset $T^1 \subset HF^1$ discussed is Section 2.

Following Abraham and Marsden [10] we use the notation

$$\Pi(X) := \hat{J}(X) \tag{3.7}$$

and refer to $\Pi(X)$ as the **n-momentum corresponding to** X. The value of the n-momentum $\Pi(X)$ at a linear frame $u = (m, e_i)$ is

$$\Pi(X)(u) = e^i(X(\pi_{LM}(u))r_i \tag{3.8}$$

which gives the $\mathbb{R}^n$ components of X with respect to the linear frame frame $u = (m, e_i)$.

These ideas can be related to the standard notions on T^*M associated with *Diff(M)*. We first show that one may use the n-symplectic action of *Diff(M)* on LM to induce the standard symplectic action of *Diff(M)* on the symplectic manifold $(T^*M, d\hat{\vartheta})$. We again consider the cotangent bundle as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$ so that points in T^*M are equivalence classes $[(u, \alpha)]$ for $u \in LM$ and $\alpha \in \mathbb{R}^{n*}$. We use the n-symplectic action $\tilde{\Phi}$: *Diff(M)* $\times LM \to LM$ to induce a left action $\hat{\Phi}$: *Diff(M)* $\times T^*M \to T^*M$ by defining

$$\hat{\Phi}(f, [(u, \alpha)]) = [\tilde{\Phi}(f, u), \alpha] \quad . \tag{3.9}$$

Because $\tilde{\Phi}$ is a left action on LM it is easy to see that this definition is independent of choice of representatives. Using the identification $[(u, \alpha)] \longrightarrow u(\alpha) = (m, e^i\alpha_i)$ for $u = (m, e_i)$ one can show that this action on T^*M is the standard symplectic action associated with *Diff(M)* (see, for example, [10], page 283).

The n-symplectic momentum mapping discussed above can be used to induce the momentum mapping on T^*M associated with *Diff(M)* as follows. For each vector field X on M define the map $P(X) : T^*M \to \mathbb{R}$ by

$$P(X)([u, \alpha]) :=< \Pi(X)(u), \alpha > \tag{3.10}$$

where the brackets now denote the natural inner product of elements of $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. It is not difficult to show that $P(X)$ defined here is the momentem of X as defined on page 283 in [10]. Hence the symplectic action of *Diff(M)* on T^*M and the associated momentum mapping are induced from the n-symplectic action of *Diff(M)* on LM and the n-symplectic momentum mapping defined in (3.5) above.

We provide the interpretation of conservations laws associated with n-symplectic momentum mappings. First a preliminary Lemma. Let $\hat{f} = \hat{f}^i r_i \in T^1$ and $\hat{g} = \hat{g}^{ij} r_i \otimes r_j \in ST^2$, let $X_{\hat{f}}$ be the Hamiltonian vector field of $\hat{f}$, and let $[\![X_{\hat{g}}]\!] = [\![X_{\hat{g}}]\!]^i r_i$ be the equivalence class of $\mathbb{R}^n$-valued Hamiltonian vector fields of $\hat{g}$.

LEMMA 3.1: If $\{\hat{g}, \hat{f}\} = 0$ then for each $i = 1, 2, \ldots, n$, $\hat{f}^i$ is constant on the orbits of each ${X_{\hat{g}}}^i \in [\![X_{\hat{g}}]\!]^i$.

PROOF:

Let F_t^i be the flow of ${X_{\hat g}}^i \in [\![X_{\hat g}]\!]^i$. Then

$$\begin{aligned}\frac{d}{dt}(\hat f^i \circ F_t^i) &= (F_t^i)^*(\mathbf{L}_{{X_{\hat g}}^i}{(\hat f)}^i) \\ &= (F_t^i)^*({X_{\hat g}}^i(\hat f^i)) \\ &= (F_t^i)^*((\frac{1}{2})\{\hat g, \hat f\}^{ii}) \quad .\end{aligned} \tag{3.11}$$

This vanishes iff $\{\hat f, \hat g\}^{ii} = 0$ which is true when $\{\hat f, \hat g\} = 0$. Moreover this is true for each ${X_{\hat g}}^i \in [\![X_{\hat g}]\!]^i$ since the Poisson bracket is independent of choice of representative. ▮

We now consider the situation where $\hat g \in ST^2$ is a Hamiltonian tensor, and $\hat g$ is invariant under some Lie subgroup G of *Diff(M)* (See Section 6 for an explicit example). The proof of the following theorem is given in the Appendix.

Theorem 3.1. Let Φ be an n-symplectic action of a subgroup G of *Diff(M)* on $(LM, d\theta)$ with n-momentum mapping J. Suppose $\hat g \in ST^2$, $\hat g : LM \to \mathbb{R}^n \otimes_s \mathbb{R}^n$, is invariant under the action, that is,

$$\hat g(\Phi_h(u)) = \hat g(u) \qquad \text{for all} \;\; u \in LM \;,\; h \in G \tag{3.12}$$

Then J provides n **integrals** of $\hat g$ in the sense that

$$J^i(F_t^i(u)) = J^i(u) \tag{3.13}$$

where F_t^i is the flow of any ${X_{\hat g}}^i \in [\![X_{\hat g}]\!]^i$, $i = 1, 2, \ldots, n$.

There is an obvious extension of these results to the case where $\hat g \in ST^p$ for $p \geq 3$.

4. Connections on $\beta : AM \longrightarrow LM$

In Section 2 we saw that the kernel of the map $\hat f \to X_{\hat f}$ for $\hat f \in HF^1$ is the set of constant $\mathbb{R}^n$-valued functions in the set $C^\infty(M, \mathbb{R}^n)$. Following the example [6] of geometric quantization based on symplectic geometry on T^*M we seek to lift the set of Hamiltonian vector fields HV^1 to a Lie algebra of vector field $\widetilde{HV^1}$ on a bundle over LM in such a way that we obtain a Lie algebra isomorphism between HF^1 and $\widetilde{HV^1}$. We will show that the affine frame bundle of a manifold is an appropriate bundle to accomplish this task.

Let AM denote the principal fiber bundle of affine frames [1,2,3] over an n-dimensional manifold M. A point $w \in AM$ is a triple (m, e_i, v) where (e_i) is a linear frame at $m \in M$ and v is a tangent vector at m. The vector v is the "origin" of the affine frame. The semi-direct product affine group $A(n) = GL(n) \ltimes \mathbb{R}^n$ acts on AM on the right by

$$(m, e_i, v) \cdot (g, \xi) = (m, e_i g^i_j, v + e_i \xi^i) \tag{4.1}$$

for each $(g, \xi) = ((g^i_j), \xi^i) \in A(n)$.

There is a canonical embedding $\gamma : LM \to AM$ of LM into AM given by $\gamma(m, e_i) = (m, e_i, 0)$, and an associated canonical projection mapping $\beta : AM \to LM$ given by $\beta(m, e_i, v) = (m, e_i)$. The existence of the maps γ and β implies [1] that AM is a trivial principal $\mathbb{R}^n$ bundle over LM.

In the following it will be convenient to have available the following canonical coordinates on AM. Let (x^i) be a coordinate chart on $U \subset M$, and define canonical coordinates (x^i, π^j_k) on $\hat{U} = \pi^{-1}(U) \subset LM$ as in (2.2). On $\tilde{U} = \beta^{-1}(\hat{U}) \subset AM$ define canonical coordinates (x^i, π^j_k, y^a) by

$$\begin{aligned} x^i(m, e_i, v) &= x^i(m) \\ \pi^j_k(m, e_i, v) &= e^j(\frac{\partial}{\partial x^k}) \\ y^a(m, e_i, v) &= e^a(v) \quad . \end{aligned} \tag{4.2}$$

Note that the coordinates y^a on AM are globally defined.

The bundles LM and AM support two invariantly defined forms. The $\mathbb{R}^n$-valued soldering one-form $\theta = \theta^i r_i$ on LM was defined above in (2.3). In the canonical coordinates (2.2) θ has the local coordinate representation

$$\theta = (\pi^i_j dx^j) r_i \quad . \tag{4.3}$$

We introduce here the $\mathbb{R}^n$-valued **canonical zero-form** $\lambda = \lambda^i r_i$ on AM defined by

$$\lambda(m, e_i, v) = e^i(v) r_i \quad . \tag{4.4}$$

It is evident from (4.2) and (4.4) that λ has the local coordinate representation

$$\lambda = y^i r_i \quad . \tag{4.5}$$

In the following we will need the particular connection on the principal $\mathbb{R}^n$ bundle $AM \xrightarrow{\beta} LM$ given in the following Lemma.

Lemma 4.1: The $\mathbb{R}^n$-valued one-form $\sigma = \beta^*\theta + d\lambda$ is a connection on the bundle $AM \xrightarrow{\beta} LM$. The curvature Σ of σ is the $\mathbb{R}^n$-valued two-form

$$\Sigma = d\sigma = \beta^*(d\theta) \quad . \tag{4.6}$$

We will also need the fundamental vertical vector fields on AM. For each $\xi = \xi^i r_i$ in the Lie algebra $\mathbb{R}^n$ of the group $\mathbb{R}^n$ let η_ξ denote the associated fundamental vertical vector field on AM. The explicit coordinate representation for η_ξ is

$$\eta_\xi = \xi^i \frac{\partial}{\partial y^i} \quad . \tag{4.7}$$

Next suppose that $\hat{f} = \hat{f}^i r_i : LM \to \mathbb{R}^n$. For each such function we define a vertical vector field $\eta_{\hat{f}}$ on AM by the formula

$$\eta_{\hat{f}}(w) = \eta_{\hat{f}(\beta(w))}(w) = (\hat{f}^i(\beta(w))\frac{\partial}{\partial y^i} \quad . \tag{4.8}$$

Let X be a vector field on AM. Then X is horizontal with respect to the connection σ iff $\sigma(X) = 0$. The local coordinate expression for such a horizontal vector field is

$$X = A^i \frac{\partial}{\partial x^i} + B^i_j \frac{\partial}{\partial \pi^i_j} - \pi^i_j A^j \frac{\partial}{\partial y^i} \quad . \tag{4.9}$$

If $X = A^i \frac{\partial}{\partial x^i} + B^i_j \frac{\partial}{\partial \pi^i_j}$ is a vector field on LM then its horizontal lift $X^{\#}$ to AM with respect to σ is given by (4.9) where the components A^i and B^i_j are pull-ups under β of functions defined on LM. Finally we recall that the vertical part of a tangent vector X at $w \in AM$ may be expressed as

$$ver(X) = \eta_{\sigma(X)} \equiv \eta_{(X \;\lrcorner\; \sigma)} \quad . \tag{4.10}$$

We now wish to characterize the vector fields on AM that preserve the connection one-form σ introduced above. The techniques used here follow those used by Sniatycki [6]. Let ζ be a vector field on AM such that $L_\zeta \sigma = 0$. Expanding this equation we have

$$\zeta \;\lrcorner\; d\sigma + d(\zeta \;\lrcorner\; \sigma) = 0 \quad . \tag{4.11}$$

Evaluating this on the fundamental vertical vector field η_ξ we find

$$\eta_\xi(\zeta \;\lrcorner\; \sigma) = 0 \tag{4.12}$$

since the curvature $d\sigma$ is horizontal. Hence the $\mathbb{R}^n$-valued function $\zeta \;\lrcorner\; \sigma$ is constant on fibers of AM so that we may express it as

$$\zeta \;\lrcorner\; \sigma = \hat{f} \circ \beta \tag{4.13}$$

for some $\mathbb{R}^n$-valued function $\hat{f}$ on LM. From (4.10) and (4.13) we find that the vertical component $ver(\zeta)$ is given by

$$ver(\zeta) = \eta_{\hat{f}} \quad . \tag{4.14}$$

Using (4.13) and (4.14) back in (4.11) with $\zeta = ver(\zeta) + hor(\zeta)$ where $hor(\zeta)$ denotes the horizontal part of ζ we find

$$\begin{aligned} d(\hat{f} \circ \beta) &= -hor(\zeta) \;\lrcorner\; d\sigma \\ &= -hor(\zeta) \;\lrcorner\; \beta^*(d\theta) \quad . \end{aligned} \tag{4.15}$$

Hence $hor(\zeta)$ is the horizontal lift $X_{\hat{f}}{}^{\#}$ to AM of the vector field $X_{\hat{f}}$ on LM determined by the n-symplectic structure equation

$$d\hat{f} = -X_{\hat{f}} \;\lrcorner\; d\theta \quad . \tag{4.16}$$

The result is the following. The set $\widetilde{HV^1}$ of all vector fields ζ on AM that preserve the connection σ is a Lie algebra under Lie bracket. If $\zeta \in \widetilde{HV^1}$ then

$$\zeta = \zeta_{\hat{f}} = X_{\hat{f}}{}^{\#} + \eta_{\hat{f}} \tag{4.17}$$

for some $\mathbb{R}^n$-valued function $\hat{f}$ on LM, where $X_{\hat{f}}$ is determined by (4.16). Now equation (4.16) is the n-symplectic structure equation on $(LM, d\theta)$, and from the general theory we know that the set of $\mathbb{R}^n$-valued functions on LM that is compatible with (4.16) is the subset $HF^1 = T^1 \oplus C^\infty(M, \mathbb{R}^n)$ of $C^\infty(LM, \mathbb{R}^n)$. Moreover, by direct calculation one shows that $[\zeta_{\hat{f}}, \zeta_{\hat{g}}] = \zeta_{\{\hat{f},\hat{g}\}}$.

Theorem 4.1. The set of vector fields on AM that preserve the connection σ is composed of vector fields of the form (4.17) for $\hat{f} \in HF^1$. Moreover, the map

$$\hat{f} \longrightarrow \zeta_{\hat{f}} \tag{4.18}$$

for $\hat{f} \in HF^1$ defines a Lie algebra isomorphism between $(HF^1, \{\ ,\ \})$ and $(\widetilde{HV^1}, [\ ,\ ])$. $\blacksquare$

Finally we consider the locally defined position and momentum variables $\hat{x}^i = x^i r_i$ (no sum on i) and $\hat{\pi}_j = \pi^k_j r_k$. From (2.14) with $p = 1$ we obtain the associated Hamiltonian vector fields

$$X_{\hat{x}^i} = -\frac{\partial}{\partial \pi^i_i} \quad , \quad X_{\hat{\pi}_j} = \frac{\partial}{\partial x^j} \quad . \tag{4.19}$$

Using (2.8) we have $\{\hat{\pi}_i, \hat{x}^j\} = \delta^j_i r_i$. Then using (4.19) together with (4.8) we find

$$\zeta_{\hat{x}^i} = -\frac{\partial}{\partial \pi^i_i} + x^i \frac{\partial}{\partial y^i} \quad , \quad \zeta_{\hat{\pi}_j} = \frac{\partial}{\partial x^j} \quad . \tag{4.20}$$

Following [6] we define the associated prequantization operators by

$$\mathcal{P}_{\hat{x}^i} = -i\hbar\zeta_{\hat{x}^i} \quad , \quad \mathcal{P}_{\hat{\pi}_j} = -i\hbar\zeta_{\hat{\pi}_j} \quad . \tag{4.21}$$

Defining the quantum commutator as the negative of the Lie bracket of vector fields, $[\mathcal{P}_{\hat{f}}, \mathcal{P}_{\hat{g}}]_{\mathcal{Q}} = -[\mathcal{P}_{\hat{f}}, \mathcal{P}_{\hat{g}}]$, we now find

$$[\mathcal{P}_{\hat{\pi}_j}, \mathcal{P}_{\hat{x}^i}]_{\mathcal{Q}} = i\hbar\mathcal{P}_{\{\hat{\pi}_i, \hat{x}^j\}} \quad , \tag{4.22}$$

which are the quantum canonical commutation relations.

5. $L^\times \to T^*M$ as a Fiber Bundle Associated to AM

Let $\mathbb{C}^\times$ denote the non-zero complex numbers c, and denote elements of $\mathbb{R}^{n*}$ by $\alpha = (\alpha_j)$. Then it is straighforward to check that the mapping $\Phi : A(n) \times (\mathbb{R}^{n*} \times \mathbb{C}^\times) \longrightarrow \mathbb{R}^{n*} \times \mathbb{C}^\times$ defined by

$$\Phi((g,\xi),(\alpha,c)) = ((g^{-1})^i_j \alpha_i, c \cdot exp(2\pi i (g^{-1})^i_j \xi^j \alpha_i) \quad , \quad \forall\ (g,\xi) \in A(n) \tag{5.1}$$

is a left action. Using this left action we can define an associated bundle in the standard way, namely

$$E := AM \times_{A(n)} (\mathbb{R}^{n*} \times \mathbb{C}^{\times}) \quad . \tag{5.2}$$

Points in this bundle, as a bundle over the base space M, are equivalence classes

$$[((m, e_i, v), (\alpha_i, c))] \tag{5.3}$$

with equivalence defined as follows:

$$((m, e_i, v), (\alpha_i, c)) \sim ((m, e_i, v) \cdot (h, \xi), (h, \xi)^{-1} \cdot (\alpha_i, c)) \tag{5.4}$$

for all $(h, \xi) \in A(n)$. Working out the right-hand member using (5.1) we find

$$((m, e_i, v), (\alpha_i, c)) \sim ((m, e_i h^i_j, v + e_i \xi^i), (h^i_j \alpha_i, c \cdot exp(-2\pi i \xi^a \alpha_a)) \quad . \tag{5.5}$$

Note that only the $GL(n) \subset A(n)$ element (h^i_j) acts on the $\mathbb{R}^{n*}$ factor, and that only the $\mathbb{R}^n \subset A(n)$ element (ξ^i) acts on the $\mathbb{C}^{\times}$ factor. We can use this fact to show that this bundle E can be viewed as a principal $\mathbb{C}^{\times}$ bundle over the cotangent bundle.

Let $L^{\times} = L^{\times}(T^*M, \mathbb{C}^{\times})$ denote the trivial $\mathbb{C}^{\times}$ principal fiber bundle over the cotangent bundle. Define a map $\rho : E \longrightarrow L^{\times}$ as follows:

$$\rho([((m, e_i, v), (\alpha_i, c))]) = ((m, e^i \alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a)) \quad . \tag{5.6}$$

The range is clearly correct and we need only check that the image of an equivalence class is independent of choice of representative. Using (5.5) we find

$$\begin{aligned}
\rho([(m, &e_i h^i_j, v + e_i \xi^i), (h^i_j \alpha_i, c \cdot exp(-2\pi i \xi^a \alpha_a)]) \\
&= ((m, e^i (h^{-1})^j_i (h^a_j \alpha_a)), c \cdot exp(-2\pi i \xi^a \alpha_a) \cdot exp(2\pi i (h^{-1})^j_a e^a (v + e_b \xi^b) h^k_j \alpha_k)) \\
&= ((m, e^i \alpha_i), c \cdot exp(2\pi i [-\xi^a \alpha_a + e^a (v + e_k \xi^k) \alpha_a])) \\
&= ((m, e^i \alpha_i), c \cdot exp(2\pi i [-\xi^a \alpha_a + e^a (v) \alpha_a + \xi^a \alpha_a])) \\
&= ((m, e^i \alpha_i), c \cdot exp(2\pi i e^a (v) \alpha_a)) \quad .
\end{aligned} \tag{5.7}$$

Hence the map ρ is well-defined and the associated bundle E is bundle isomorphic to the trivial $\mathbb{C}^{\times}$ principal bundle of T^*M, and from now on we will consider E as composed of pairs $((m, e^i \alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a))$ and identify E with $L^{\times}$.

In the last section we saw that the connection $\sigma = \beta^*(\theta) + d\lambda$ on the principal bundle $\beta : AM \to LM$ provides a Lie algebra of vector fields $\zeta_{\hat{f}}$ on AM that leave σ invariant. This Lie algebra is isomorphic to the Lie algebra of rank 1 Hamiltonian functions HF^1 on LM under the Poisson bracket $\{\ ,\ \}$. This isomorphism is analogous to the isomorphism of the Lie algebra of vector fields $\tilde{\zeta}_f$ on $L^{\times}$ that leave the connection $\tilde{\sigma} = \pi^*(\vartheta) + \frac{1}{2\pi i}\frac{dz}{z}$ on $L^{\times}$ fixed with the Lie algebra of smooth functions on T^*M under Poisson bracket. What we show here is that the connection $\tilde{\sigma}$ on $L^{\times}$ can be induced from the connection σ on AM. We will also obtain a mapping of the subalgebra of vector fields $\zeta_{\hat{f}}$ on AM defined by elements $\hat{f} \in HF^1$ onto the subalgebra of vector fields $\tilde{\zeta}_f$ on $L^{\times}$ for f a linear polynomial observable on T^*M.

Note first that there are just two orbits of GL(n) on $\mathbb{R}^{n*}$, namely $\{0\}$ and $\mathbb{R}^{n*} - \{0\}$. There is a natural projection $AM \times (\mathbb{R}^{n*} \times \mathbb{C}^\times) \longrightarrow E$ given by $(u, (\alpha_i, c)) \longrightarrow u(\alpha_i, c)$ where $u(\alpha_i, c)$ denotes the equivalence class (5.3) above. Fix a a non-zero element $\alpha \in \mathbb{R}^{n*}$ and an element $c \in \mathbb{C}^\times$ and define a map $\psi_{(\alpha,c)} : AM \longrightarrow E \equiv L^\times$ by

$$\psi_{(\alpha,c)}(u) = u((\alpha, c)) \quad . \tag{5.8}$$

Since $\alpha \neq 0$ the maps $\psi_{(\alpha,c)}$, for $c \in \mathbb{C}^\times$, map AM onto all of $L^\times$ except for the section S_0 that contains the zero section of T^*M. This section is a closed subset of $L^\times$.

For each $q \in L^\times$ define [1] the vertical subspace at q as the subspace of vectors tangent to the $\mathbb{C}^\times$ fiber through q. We map the horizontal spaces H_u on AM to horizontal spaces $\tilde{H}_q$ on $L^\times$ as follows. For each $q \in L^\times - S_0$ choose a $u \in AM$ and a pair $(\alpha_i, c) \in \mathbb{R}^{n*} \times \mathbb{C}^\times$ with $\alpha \neq 0$ such that $\psi_{(\alpha,c)}(u) = q$. Define the horizontal space $\tilde{H}_q$ as the image of H_u under the map $\psi_{(\alpha,c)}$, namely

$$\tilde{H}_q = \psi_{(\alpha,c)*}(H_u) \quad . \tag{5.9}$$

We calculate $\tilde{H}_q$ using local coordinates.

H_u is the set of all tangent vectors X at $u \in AM$ such that $\sigma(X) = 0$, and the local coordinate form of such vectors is given in (4.9) above. Let $q = [((m, e_i, v), (\alpha_i, c))] = ((m, e^i\alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a))$ using the identification (5.6) above. On $L^\times$ we use the local coordinates (x^i, p_j, z) defined by

$$\begin{aligned} x^i(((m, e^i\alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a)) &= x^i(m) \quad , \\ p_j(((m, e^i\alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a)) &= e^i\alpha_i(\frac{\partial}{\partial x^j}) \quad , \\ z(((m, e^i\alpha_i), c \cdot exp(2\pi i e^a(v)\alpha_a)) &= c \cdot exp(2\pi i e^a(v)\alpha_a) \quad . \end{aligned} \tag{5.10}$$

For $X \in H_u$ given in (4.9) write

$$\tilde{X} = \psi_{(\alpha,c)*}(X) = M^i\frac{\partial}{\partial x^i} + N_i\frac{\partial}{\partial p_i} + Q\frac{\partial}{\partial z} \quad . \tag{5.11}$$

Evaluating (5.11) using (4.9) and (5.8) we find that H_q, for q not in S_0, is composed of vectors of the form

$$\tilde{X} = A^i\frac{\partial}{\partial x^i} + B^i_j\alpha_i\frac{\partial}{\partial p_j} - 2\pi i A^j p_j(q)z(q)\frac{\partial}{\partial z} \quad . \tag{5.12}$$

Since $\alpha \neq 0$ and the B^i_j are arbitrary, $B^i_j\alpha_i$ may be considered as n arbitrary constants which we denote by B_j. Hence H_q, for q not in S_0, is composed of vectors of the form

$$\tilde{X} = A^i\frac{\partial}{\partial x^i} + B_j\frac{\partial}{\partial p_j} - 2\pi i A^j p_j(q)z(q)\frac{\partial}{\partial z} \quad . \tag{5.13}$$

LEMMA 5.1. The distribution H of subspaces H_q on $L^\times - S_0$ defined by tangent vectors of the form (5.13) is invariant under right translation by $\mathbb{C}^\times$ on $L^\times - S_0$.

PROOF: The Lemma follows easily upon noting that $R_{c_1} \circ \psi_{(\alpha,c)} = \psi_{(\alpha,c_1c)}$. ▮

It follows from (5.13) that H defines a complement to the vertical spaces at points of $L^\times - S_0$. Now if the distribution H defined above on $L^\times - S_0$ were defined on all of $L^\times$ then it would define a connection. We will use H to define a connection one-form on $L^\times - S_0$ and then extend it to all of $L^\times$ by continuity.

Thus let $\tilde{\sigma}$ be a one-form on $L^\times$ that has the properties of a connection, namely $\tilde{\sigma}(c^*) = c$ and $R_c^*(\tilde{\sigma}) = \tilde{\sigma}$. Here c^* is the fundamental vertical vector field determined by $c \in \mathbb{C}$, where we consider $\mathbb{C}$ as the Lie algebra of $\mathbb{C}^\times$ under the identification $c \longrightarrow exp(2\pi ic)$. Then c^* is given by $c^* = 2\pi icz\frac{\partial}{\partial z}$. If $\tilde{\sigma}$ satisfies the above two conditions it must be of the form

$$\tilde{\sigma} = \pi^*(\mu) + \frac{1}{2\pi i}\frac{dz}{z} \tag{5.14}$$

where μ is a real-valued one-form on T^*M, and where $\pi : L^\times \to T^*M$ is the projection map. Expressing $\tilde{\sigma}$ in local coordinates (x^i, p_j, z) we have

$$\tilde{\sigma} = R_i dx^i + S^i dp_i + \frac{1}{2\pi i}\frac{dz}{z} \tag{5.15}$$

where R_i and S^i are pull-ups of functions defined on T^*M.

We now require $\tilde{\sigma}$ to also satisfy $\tilde{\sigma}(X) = 0$ on $L^\times - S_0$ for X of the form given in (5.13). We find $R_i = p_i$ and $S^i = 0$, and thus our desired one-form on $L^\times - S_0$ has the local coordinate form

$$\tilde{\sigma} = p_i dx^i + \frac{1}{2\pi i}\frac{dz}{z} \tag{5.16}$$

The invariant form of $\tilde{\sigma}$ on $L^\times - S_0$ is then

$$\tilde{\sigma} = \pi^*(\vartheta) + \frac{1}{2\pi i}\frac{dz}{z} \tag{5.17}$$

By continuity we can extend this one-form to all of $L^\times$ so that (5.17) gives the desired connection one-form. We have shown:

Theorem 5.1 The connection one-form defined on $L^\times$ by the distribution H that is induced by the horizontal distribution of the connection one-form $\sigma = \beta^*(\theta) + d\lambda$ on $\beta : AM \to LM$ is $\tilde{\sigma} = \pi^*(\vartheta) + \frac{1}{2\pi i}\frac{dz}{z}$.

Consider the Lie algebra of vector fields $\zeta_{\hat{f}}$ on AM that satisfy $L_{\zeta_{\hat{f}}}\sigma = 0$. These vector fields where characterized in Section 4 and are of the form $\zeta_{\hat{f}} = X^{\#}_{\hat{f}} + \eta_{\hat{f}}$ for $\hat{f} \in HF^1$ on LM, where $X_{\hat{f}}$ is determined by the n-symplectic structure equation on LM, and where

$$\eta_{\hat{f}} = \hat{f}^i \circ \beta \frac{\partial}{\partial y^i} \tag{5.18}$$

is a vertical vector field. It is clear that the vector fields $\zeta_{\hat{f}}$ have the same form as the vector fields $\tilde{\zeta}_f$ on $L^\times$ provided $f : T^*M \to \mathbb{R}$ is a linear polynomial observable on T^*M. We know that the homogenous part of such observables are uniquely related to the momentum mapping defined by *Diff(M)*, and that the homogeneous part of $\hat{f}$ on LM is also uniquely related to the n-momentum mapping on LM determined by *Diff(M)*. The question is whether or not the vector fields on $L^\times$ can be obtained from the vector fields on AM. First we consider the maps. Fixing the pair (α, c) with $\alpha \neq 0$ we consider the mapping $\psi_{(\alpha,c)} : AM \to L^\times$. The following Lemma, which characterizes the many-to-one nature of the mappings $\psi_{(\alpha,c)}$, follows easily from the definition (5.8).

LEMMA 5.2. The invariance group of the mapping $\psi_{(\alpha,c)} : AM \to L^\times$ for a fixed pair (α, c) with $\alpha \neq 0$ is the subgroup $G_{(\alpha,c)} \subset A(n)$ defined by

$$G_{(\alpha,c)} = \{(g^i_j, w^k) \mid g^i_j\alpha_i = \alpha_j \ , \ w^i\alpha_i = n = 0, \pm 1, \pm 2, \dots \} \tag{5.19}$$

We now consider the mapping of vectors $\zeta_{\hat{f}}$ on $\beta : AM \to LM$ to $L^\times - S_0$. We note first that the horizontal lift $X^{\#}_{\hat{f}}$ on AM of a vector field $X_{\hat{f}}$ on LM is invariant under right translation: $R_{\xi *}(X^{\#}_{\hat{f}}) = X^{\#}_{\hat{f}}$. Moreover, a vector field of the form $\hat{f}^i \circ \beta \frac{\partial}{\partial y^i}$ satisfies

$$R_{\xi *}(\hat{f}^i \circ \beta(u)\frac{\partial}{\partial y^i}(u)) = \hat{f}^i \circ \beta(u) R_{\xi *}(\frac{\partial}{\partial y^i}(u)) = \hat{f}^i \circ \beta(u\cdot\xi)(\frac{\partial}{\partial y^i}(u\cdot\xi)) \tag{5.20}$$

since the vertical basis vectors are themselves invariant under right translations by elements $\xi \in \mathbb{R}^n$. The vertical vector fields $\eta_{\hat{f}}$ are thus invariant by right translation on $\beta : AM \to LM$. The result is that $R_{\xi *}(\zeta_{\hat{f}}) = \zeta_{\hat{f}}$.

Now once again fix an element $(\alpha, c) \in \mathbb{R}^{n*} \times \mathbb{C}^\times$ with $\alpha \neq 0$. For each $\zeta_{\hat{f}}$ we consider the set of vectors at points in $L^\times_{(\alpha,c)} \subset L^\times - S_0$ determined by (α, c) defined by

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}) \quad . \tag{5.21}$$

Since $\psi_{(\alpha,c)}$ is C^∞ (5.21) will define a smooth vector field on $L^\times_{(\alpha,c)}$ provided the many-to-one map $\psi_{(\alpha,c)}$ defines unique vectors on $L^\times_{(\alpha,c)}$.

Let $u, \bar{u}$ be two points on a common fiber in AM with $\bar{u} = R_{(I,\xi)}(u)$ with $(I, \xi) \in G_{(\alpha,c)}$. Then

$$\psi_{(\alpha,c)}(\bar{u}) = \psi_{(\alpha,c)} \circ R_{(I,\xi)}(u) = \psi_{(\alpha,c)}(u) \quad . \tag{5.22}$$

Using $R_{\xi *}(\zeta_{\hat{f}}) = \zeta_{\hat{f}}$ and (5.22) one can show that

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}(\bar{u}) = \psi_{(\alpha,c)*}(\zeta_{\hat{f}}(u)) \tag{5.23}$$

when $\bar{u} = R_{(I,\xi)}(u)$ with $(I, \xi) \in G_{(\alpha,c)}$. Hence each vector field $\zeta_{\hat{f}}$ does indeed define a vector field on $L^\times_{(\alpha,c)}$. Since we must use a different map $\psi_{(\alpha,c)}$ for each c, we need to check that we get unique tangent vectors as c varies. We will show, by looking at the local coordinate formula for $\psi_{(\alpha,c)*}(\zeta_{\hat{f}})$, that we get unique tangent vectors and that they may be extended from the open submanifolds $L^\times_{(\alpha,c)}$ to include the closed subset S_0.

Now each $\zeta_{\hat{f}}$ has the form given in (4.1) above, and the image of the horizontal part $X^{\#}_{\hat{f}}$ has the form given in (5.12). We use (5.12) rather than (5.13) here because we will rewrite the middle term when X has the special form of a $\zeta_{\hat{f}}$. Using (5.18) one shows that

$$\psi_{(\alpha,c)*}(\eta_{\hat{f}}) = 2\pi i(\hat{f}^j \circ \psi_{(\alpha,c)})\alpha_j z\frac{\partial}{\partial z} \quad . \tag{5.24}$$

Hence at points of $L^{\times}_{(\alpha,c)}$ we have that :

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}) = \left(A^i\frac{\partial}{\partial x^i} + B^i_j\alpha_i\frac{\partial}{\partial p_j} - 2\pi i A^j p_j(q)z(q)\frac{\partial}{\partial z}\right) + \left(2\pi i(\hat{f}^j \circ \psi_{(\alpha,c)})\alpha_j z\frac{\partial}{\partial z}\right) . \tag{5.25}$$

There are two cases to consider.

<u>**Case I:** $\hat{f} \in T^1 \subset HF^1$**:**</u>

In this case $\hat{f}$ and $X_{\hat{f}}$ on LM (see (2.14)) and $\zeta_{\hat{f}}$ on AM are given in local coordinates by

$$\begin{aligned}
\hat{f} &= f^i(x)\pi^j_i r_j \\
X_{\hat{f}} &= f^i(x)\frac{\partial}{\partial x^i} - \frac{\partial f^i}{\partial x^j}\pi^k_i\frac{\partial}{\partial \pi^k_j} \\
\zeta_{\hat{f}} &= f^i(x)\frac{\partial}{\partial x^i} - \frac{\partial f^i}{\partial x^j}\pi^k_i\frac{\partial}{\partial \pi^k_j}
\end{aligned} \tag{5.26}$$

Using these results in (5.25) we have for $\hat{f} \in T^1$:

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}) = \left(f^i(x)\frac{\partial}{\partial x^i} - \frac{\partial f^a}{\partial x^j}\pi^i_a\alpha_i\frac{\partial}{\partial p_j} - 2\pi i f^j(x)p_j z\frac{\partial}{\partial z}\right) + \left(2\pi i f^k(x)\pi^j_k\alpha_j z\frac{\partial}{\partial z}\right) . \tag{5.27}$$

Since $\pi^j_k(m, e_i)\alpha_j = p_k(m, e^i\alpha_i)$ the vertical components in this last equation cancel. Moreover, the remaining parts are smooth on all of $L^{\times}$ and are independent of choice of c. Hence we have:

LEMMA 5.3. For $\hat{f} \in T^1 \subset HF^1$ the vector field $\psi_{(\alpha,c)*}(\zeta_{\hat{f}})$ on $L^{\times}$ determined by $\zeta_{\hat{f}}$ on AM has the local coordinate form

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}) = f^i(x)\frac{\partial}{\partial x^i} - \frac{\partial f^a}{\partial x^j}p_a\frac{\partial}{\partial p_j} \tag{5.28}$$

REMARK: This vector field is the vector field ζ_f on $L^{\times}$ that one obtains from X_f on T^*M when $f \in C^\infty(T^*M, \mathbb{R})$ is the momentum $P(\vec{f})$ of a vector field $\vec{f} = f^i(x)\frac{\partial}{\partial x^i}$ on M.

<u>**Case II:** $\hat{f} \in C^\infty(M, \mathbb{R}^n)$**.**</u>

In this case $\hat{f}$ and $X_{\hat{f}}$ on LM (see (2.14)) and $\zeta_{\hat{f}}$ on AM are given in local coordinates by

$$\begin{aligned}\hat{f} &= f^i(x) r_i \\ X_{\hat{f}} &= -\frac{\partial f^i}{\partial x^j}\frac{\partial}{\partial \pi^i_j} \\ \zeta_{\hat{f}} &= -\left(\frac{\partial f^i}{\partial x^j}\frac{\partial}{\partial \pi^i_j}\right) + \left(f^i(x)\frac{\partial}{\partial y^i}\right)\end{aligned} \tag{5.29}$$

Using this result back in (5.25) we find

$$\psi_{(\alpha,c)}(\zeta_{\hat{f}}) = \left(-\frac{\partial f^i}{\partial x^j}\alpha_i\frac{\partial}{\partial p_j}\right) + \left(2\pi i f^j \alpha_j z\frac{\partial}{\partial z}\right) \quad . \tag{5.30}$$

Now we recall that we have fixed $\alpha \neq 0$. Hence $f^i(x)\alpha_i$ is, for each $\hat{f} = f^i(x)r_i$, a real-valued function on $L^\times$ that is constant on fibers. We introduce the notation

$$f_\alpha(x) = f^i(x)\alpha_i \tag{5.31}$$

for such functions. Then we may rewrite (5.30) as

$$\psi_{(\alpha,c)}(\zeta_{\hat{f}}) = \left(-\frac{\partial f_\alpha}{\partial x^j}\frac{\partial}{\partial p_j}\right) + \left(2\pi i f_\alpha z\frac{\partial}{\partial z}\right) \quad . \tag{5.32}$$

This is smooth on all of $L^\times$ and independent of choice of c. We have the result:

LEMMA 5.4. For $\hat{f} \in C^\infty(M, \mathbb{R}^n) \subset HF^1$ the vector field $\psi_{(\alpha,c)*}(\zeta_{\hat{f}})$ on $L^\times$ determined by a fixed non-zero $\alpha \in \mathbb{R}^{n*}$ and $\zeta_{\hat{f}}$ on AM has the local coordinate form

$$\psi_{(\alpha,c)*}(\zeta_{\hat{f}}) = \left(-\frac{\partial f_\alpha}{\partial x^j}\frac{\partial}{\partial p_j}\right) + \left(2\pi i f_\alpha z\frac{\partial}{\partial z}\right) \tag{5.33}$$

where $f_\alpha(x) = f^i(x)\alpha_i$.

REMARK: This vector field is the vector field ζ_f on $L^\times$ that one obtains from X_f on T^*M when $f \in C^\infty(T^*M, \mathbb{R})$ is the function $\tau^*(f^i(x)\alpha_i)$, where τ is the projection $\tau : T^*M \to M$.

Finally in this section we show that the equivalence class $[\![X_{\hat{f}}]\!]$ of Hamiltonian vector fields determined by $\hat{f} \in ST^p$ can be mapped to the Hamiltonian vector field $X_{\tilde{f}}$ on T^*M determined by $\tilde{f} : T^*M \to \mathbb{R}$, where $\tilde{f}$ is the homogeneous degree p polynomial observable on T^*M induced by $\hat{f}$ as in (2.20). Consider T^*M as the associated bundle $LM \times_{GL(n)} \mathbb{R}^{n*}$, and fix a non-zero $\alpha = (\alpha_i) \in \mathbb{R}^{n*}$. Then define the map $\psi_\alpha : LM \to T^*M - \tilde{S}_0$, where $\tilde{S}_0$ is the zero section of T^*M , by

$$\psi_\alpha(u) = u(\alpha) = [u, \alpha] \quad . \tag{5.34}$$

This map, like the map $\psi_{(\alpha,c)}$ discussed above, is a many-to-one map with $\psi_\alpha(u{\cdot}h) = \psi_\alpha(u)$ for $h \in G_\alpha \subset GL(n)$ where G_α is the stability group of α.

Theorem 5.2. Let $\hat{f} \in ST^p$, let $[\![X_{\hat{f}}]\!]$ be the associated equivalence class of Hamiltonian vector fields determined by (2.11), and let $\tilde{f}$ be the degree p homogeneous polynomial observable on T^*M determined by $\hat{f}$ as in (2.20). Then

$$X = p!\psi_{\alpha *}(X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{p-1}}) \quad , \tag{5.35}$$

where $X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}$ denotes any set of representatives of $[\![X_{\hat{f}}]\!]$, is a vector field on $T^*M - \tilde{S}_0$, and $X = X_{\tilde{f}}$.

The proof of this theorem is given in the appendix. The essential points to notice are that (1) the arbitrariness in the definition of the Hamiltonian vector fields cancels out under the mapping (5.35), and (2) the many-to-one map ψ_α determines a vector field on $T^*M - \tilde{S}_0$ because of the tensorial nature of the explicitly determined part of the Hamiltonian vector fields.

6. Examples of n-momentum Mappings.

(I): Linear Momentum

Let $M = \mathbb{R}^n$, $G = \mathbb{R}^n$, and let G act on M by translations:

$$\Phi : G \times M \to M : (\mathbf{v}, \mathbf{m}) \longrightarrow \mathbf{v} + \mathbf{m} \quad . \tag{6.1}$$

The infinitesimal generator corresponding to $\xi \in \mathbb{R}^n$ is $\xi_M(m) = \xi^i \frac{\partial}{\partial x^i}$. By (3.5)-(3.7) the n-momentum on LM associated with ξ is

$$\hat{J}(\xi)(u) = \Pi(\xi)(u) = \hat{\xi}(u) = \xi^i \pi_i^j(u) r_j \tag{6.2}$$

and hence the n-momentum mapping is

$$J(u) = \pi_i^j(u) r^i \otimes r_j \quad , \tag{6.3}$$

or simply

$$J = \hat{\pi}_i \otimes r^i = \pi_i^j r^i \otimes r_j \quad . \tag{6.4}$$

Hence $J(u)$ has an interpretation as a **momentum frame** rather than simply a momentum. Note that from (6.3) with $u = (m, e_i)$ we have

$$\begin{aligned} J(u) &= \pi_i^j(u) r^i \otimes r_j \\ &= e^j(\frac{\partial}{\partial x^i}) r^i \otimes r_j \ c \ . \end{aligned} \tag{6.5}$$

Thus the $\mathbb{R}^{n*} \otimes \mathbb{R}^n$ components $J(u)_i^j = e^j(\frac{\partial}{\partial x^i})$ of J at $u \in LM$ give the components of the linear frame (e_i) with respect to the coordinated linear frame $(\frac{\partial}{\partial x^i})$.

(II): Angular Momentum

Let $M = \mathbb{R}^n$ and let $G = SO(p,q)$. Let G act on M by:

$$\Phi : G \times M \to M : (T, \mathbf{m}) \longrightarrow T\mathbf{m} \quad . \tag{6.6}$$

The infinitesimal generator corresponding to $B = B^i_j E^j_i \in \mathcal{G} \subset L(\mathbb{R}^n, \mathbb{R}^n)$ is $B_M(m) = B^i_j x^j(m) \frac{\partial}{\partial x^i}$ where (E^i_j) is a basis of $\mathcal{G}$. Then from (3.5)-(3.7) the n-momentum associated with B is

$$\hat{J}(B)(u) = \Pi(B)(u) = \hat{B} = B^i_j x^j(m) \pi^k_i(u) r_k \quad , \tag{6.7}$$

and the n-momentum mapping is

$$J(u) = (x^j(m)\pi^k_i(u)) C^i_j \otimes r_k \tag{6.8}$$

where (C^i_j) is the basis of $\mathcal{G}^*$ dual to the basis (E^i_j).

Take the special case when n=4 and $G = SO(1,3)$ with M equipped with the Minkowski metric tensor $g = \eta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. Then for $B \in so(1,3)$ we have

$$B_M = B^i_j x^j \frac{\partial}{\partial x^i} \quad , \tag{6.9}$$

which is a Killing vector field on M. $\hat{J}(B)(u)$ then gives the components of B_M with respect to the linear frame $u = (m, e_i)$. Since the C^i_j are η^{ij} skew symmetric, the 4-momentum (6.8) can be written as

$$\begin{aligned} J &= (x^j \pi^k_i) \eta_{jm} C^{im} \otimes r_k \\ &= (1/2)(x_m \pi^k_i - x_i \pi^k_m) C^{im} \otimes r_k \end{aligned} \tag{6.10}$$

which has the form of a generalized **angular momentum**. Here $C^{im} = \eta^{ij} C^m_j$.

The explicit form of the conserved quantities follow from Theorem 3.1 upon taking $\hat{g} = \eta^{ab} \pi^i_a \pi^j_b r_i \otimes r_j \in ST^2$ as the Hamiltonian tensor on LM. $\hat{g}$ is invariant under the lifted action of SO(1,3) to LM. The Hamiltonian vector fields $X_{\hat{g}}$ are

$$X_{\hat{g}}{}^i = \eta^{ab} \pi^i_a \frac{\partial}{\partial x^b} \quad . \tag{6.11}$$

From Theorem 3.1 each J^i , $i = 0, 1, 2, 3$ is constant along the flow of $X_{\hat{g}}{}^i$ for the same i. Take the vector field $X_{\hat{g}}{}^1$. The equations for its integral curves are:

$$\dot{x}^i = \eta^{ij} \pi^1_j \quad , \tag{6.12a}$$

$$\dot{\pi}^i_j = 0 \quad . \tag{6.12b}$$

Equations (6.12) imply that $\ddot{x}^i = 0$ so that the trajectory on M is a geodesic. Let $u_0 = (m_0, e_{0k})$ be the initial condition. Then (6.12b) implies that

$$\pi^i_j(t) = \pi^i_j(0) = e^{0i}\left(\frac{\partial}{\partial x^j}\right) := h^i_j \quad . \tag{6.13}$$

Using this in (6.12a) we get

$$\dot{x}^i = \eta^{ij} h^1_j \Longrightarrow x^i(t) = x^i_0 + t\eta^{ij} h^1_j \quad . \tag{6.14}$$

The flow F_t^1 is then given in local coordinates by

$$(x^i, \pi_j^k) \circ F_t^1 = (x^i + t\eta^{ia}\pi_a^1, \pi_j^k) \quad . \tag{6.15}$$

We know that J^1 is constant along the flow of $X_{\hat{g}}{}^1$. From (6.8) and (6.15) we find

$$\begin{aligned} J^1 \circ F_t^1 &= (\pi_j^1 \circ F_t^1)(x^k \circ F_t^1)C_k^j \\ &= \pi_j^1(x^k + t\eta^{ka}\pi_a^1)C_k^j \\ &= \pi_j^1 x^k C_k^j + t\pi_j^1\eta^{ka}\pi_a^1 C_k^j \\ &= \pi_j^1 x^k C_k^j + t\pi_j^1\pi_a^1 C^{ja} \\ &= \pi_j^1 x^k C_k^j \quad . \end{aligned} \tag{6.16}$$

The second term in the next to last line vanishes since the C^{ja} are skew-symmetric. Now from (6.12a) we have $\pi_j^1 = \eta_{jk}\dot{x}^k$. Substituting this into (6.16) we find

$$J^1 \circ F_t^1 = x^{[i}\dot{x}^{j]}C_{ij} \tag{6.17}$$

which is the Lorentzian "angular momentum" of the observer along his straight line geodesic trajectory. The six independent conserved quantities correspond to the standard three angular momentum conservation laws together with the three laws giving the finite form of Lorentz boosts (see, for example [11], pp. 93-94).

7. Conclusions

The cotangent bundle T^*M of an n-dimensional manifold M is regarded as the canonical model of a symplectic manifold. This is because each cotangent bundle T^*M has an intrinsic and naturally defined symplectic two-form $d\vartheta$, where ϑ is the canonical one-form on T^*M , and every symplectic manifold "looks like" a cotangent bundle locally. What we have shown in this paper is that much, if not all, of the symplectic geometry on $(T^*M, d\vartheta)$ is induced from n-symplectic geometry on the bundle of linear frames LM of the manifold, where the n-symplectic potential on LM is the $\mathbb{R}^n$-valued soldering one-form θ. Specifically, we have shown:

(1) The soldering one-form θ induces the canonical one-form ϑ as in (1.1);

(2) Each symmetric $\otimes_s^p\mathbb{R}^n$-valued tensorial observable $\hat{f} \in ST^p$ on LM induces a degree p homogeneous polynomial observable $\tilde{f}$ on T^*M as in (2.20). Moreover, the equivalence class $[\![X_{\hat{f}}]\!]$ of $\otimes_s^{p-1}\mathbb{R}^n$-valued Hamiltonian vector fields determined by $\hat{f} \in ST^p$ maps to the Hamiltonian vector field of $\tilde{f}$ on T^*M as in (5.35);

(3) The n-symplectic action of *Diff(M)* and the associated n-symplectic momentum mapping on LM induce the symplectic action of *Diff(M)* and the associated momentum mapping on T^*M as given in (3.9) and (3.10), respectively.

Hence those features of symplectic geometry on T^*M associated with the polynomial observables may be considered as induced from the n-symplectic geometry of symmetric tensorial observables on LM. We have not shown that the symplectic geometry of arbitrary observables on T^*M is induced from n-symplectic geometry on LM, and whether or not this happens is an open question at the moment this.

On the other hand the polynomial observables on T^*M play a distinguished role in physical theories, especially in the theory of geometric quantization. In the case of geometric quantization formulated on T^*M , where M is the configuration space of a mechanical system, one uses the symplectic potential ϑ to construct a connection $\tilde{\sigma}$ on a trivial $\mathbb{C}^\times$ bundle $\pi : L^\times \to T^*M$ in order to construct the quantum operators associated with the observables on the phase space T^*M . These quantum operators may be defined in terms of the vector fields ζ_f on $L^\times$ that leave the connection $\tilde{\sigma}$ invariant. The map $f \to \zeta_f$ for $f \in C^\infty(T^*M, \mathbb{R})$ is a linear isomorphism from the Lie algebra of observables on T^*M under the Poisson bracket to the Lie algebra of vector fields ζ_f under the Lie bracket. In the applications of the general theory [6,7] the operators associated with the linear polynomial observables on T^*M play an especially important role. We have shown in this paper that a good deal of this structure can also be traced back to LM. In particular we have shown:

(4) The n-symplectic potential θ may be used to construct a connection σ (Lemma 4.1) on the bundle of affine frames $\beta : AM \to LM$, which is a trivial $\mathbb{R}^n$ principal bundle over LM. There is a linear isomorphism $\hat{f} \to \zeta_{\hat{f}}$ from the Lie algebra of $\mathbb{R}^n$-valued linear polynomial observables $\hat{f}$ on LM under the Poisson bracket to the Lie algebra of vector fields $\zeta_{\hat{f}}$ that leave the connection σ invariant (Theorem 4.1);

(5) The $\mathbb{C}^\times$ bundle $\pi : L^\times \to T^*M$ may be identified with a fiber bundle associated to AM. Moreover, the connection σ on $\beta : AM \to LM$ induces the connection $\tilde{\sigma}$ on $L^\times$ discussed above (Theorem 5.1);

(6) The vector fields $\zeta_{\hat{f}}$ on AM may be mapped onto the subalgebra of vector fields ζ_f on $L^\times$ associated with the linear polynomial observables on T^*M (Lemmas 5.3 and 5.4).

The results presented in this paper show that at least for polynomial observables one may replace canonical symplectic geometry on T^*M with n-symplectic geometry on the bundle of linear frames LM. But what is gained by replacing symplectic geometry based on the $\mathbb{R}$-valued symplectic potential ϑ with n-symplectic geometry based on the $\mathbb{R}^n$-valued n-symplectic potential θ? The answer lies in the new information one obtains from n-symplectic geometry about the relationship between observables and the associated Hamiltonian vector fields.

Canonical symplectic geometry on T^*M assigns to each polynomial observable, regardless of its degree, a single Hamiltonian vector field. On the other hand, n-symplectic geometry assigns to a p$^{\text{th}}$ degree symmetric polynomial observable $\hat{f} \in ST^p$ an equivalence class $[\![X_{\hat{f}}]\!]$ of $\otimes_s^{p-1}\mathbb{R}^n$-valued vector fields, and a representative of $[\![X_{\hat{f}}]\!]$ contains $\binom{n+p-2}{p-1}$ Hamiltonian vector fields $X_{\hat{f}}^{i_1 i_2 \ldots i_{p-1}}$. These sets of Hamiltonian vector fields associated with degree $p \geq 2$ symmetric polynomial observables provides new insights into the geometry and physics of such observables. We have shown in Theorem 5.2 that the arbitrariness in the definition of the equivalence classes determined by polynomial observables **cancels out** when they are mapped to, and collapsed into, a single Hamiltonian vector field on T^*M . The equivalences classes thus represent a type of "gauge freedom" that is not detectable by symplectic geometry on T^*M .

As a specific example consider a four-dimensional Riemannian spacetime M with metic tensor g. The free particle problem in the spacetime M is formulated in symplectic geometry on T^*M as follows. The metric tensor g defines a homogenous quadratic polynomial observable on T^*M , which we denote by $\tilde{g}$, and the free particle Hamiltonian is then $\tilde{H} = \tilde{g}$ where for simplicity we have choosen $m = 1/2$ for the mass of the particle. The dynamics of the free particle is given by the single Hamiltonian vector field $X_{\tilde{H}}$, and integration of $X_{\tilde{H}}$, with time-like initial conditions, tells us (1) that the trajectory on M of the particle is a geodesic of the Levi-Civita connection determined by g, and (2) that the particle parallel transports its four-momentum along the geodesic.

The formulation of the problem in 4-symplectic geometry on LM is as follows. The metric tensor g determines an element $\hat{g} \in ST^2$, and we take $\hat{H} = \hat{g}$ for the free particle Hamiltonian tensor. The dynamics is now specified by the equivalence class of Hamiltonian vector fields $[\![X_{\hat{H}}]\!] = [\![X_{\hat{H}}]\!]^i r_i$. A representative of the equivalence class is composed of four Hamiltonian vector fields (see 2.16)

$$X^i_{\hat{H}} = g^{ab}(x)\pi^i_a \frac{\partial}{\partial x^b} - \frac{1}{2}\left(\frac{\partial g^{ab}}{\partial x^k}\pi^i_a\pi^j_b + T^{ij}_k\right)\frac{\partial}{\partial \pi^j_k} \tag{7.1}$$

where $i = 1, 2, 3, 4$. The gauge freedom is specified by the components T^{ij}_k which must satisfy $T^{(ij)}_k = 0$ but are otherwise arbitrary. Since the structures of the Poisson algebra $(ST, \{\ ,\ \})$ and the Lie algebra of the associated equivalence classes of Hamiltonian vector fields are independent of choice of representatives we are free to set a gauge condition to select a representative set of vector fields. The gauge condition [4]

$$X^i_{\hat{H}} \lrcorner\ X^j_{\hat{H}} \lrcorner\ d\theta^k = 0 \qquad \forall\ i, j, k \tag{7.2}$$

determines a unique representative set of vector fields

$$X^i_{\hat{H}} = \hat{g}^{ij} B_j \tag{7.3}$$

where the four vector fields B_j are the standard horizontal vector fields of the Levi-Civita connection ω_g on LM determine by g. Integration of any one of the vector fields $X^i_{\hat{H}}$, with time-like initial conditions, now tells us (1) that the trajectory on M of the particle is a geodesic of the Levi-Civita connection ω_g determined by g, and (2) that the particle parallel transports a full linear frame along the geodesic with the time-like leg of the frame being the parallel transported four-momentum of the particle. The extra information that is not contained in the formulation of the problem on T^*M , namely the parallel transport of a spatial triad along a geodesic, together with the parallel transport of the four-momentum along the geodesic, provides the complete and correct description of a freely-falling, non-rotating particle.

There is more that one can say concerning these "free" systems on spacetime. On T^*M the constant energy surfaces to which $X_{\tilde{H}}$ is tangent are the surfaces on which $\tilde{H} = \tilde{g}$ are constant. The analogue on LM are the orthonormal subbundles obtained from the canonical orthonormal subbundle OM by conjugation. The vector fields in (7.3) are tangent to these subbundles, and the Hamiltonian tensor $\hat{H} = \hat{g}$ is constant on each of these subbundles. Hence the n-symplectic phase space LM can be thought of as the phase space of observers, since the dynamics described above is what is normally thought of as the dynamics of freely-falling, non-rotating inertial observers on spacetime. This interpretation is consistent with the interpretation given in Sections 3 and 6 of the n-momentum mapping associated with $Diff(m)$ as providing a momentum frames rather than simply momentum.

The new features present in n-symplectic geometry also offer the possibility of providing new insights in the theory of geometric quantization. As discussed at the end of Section 4 the Hamiltonian vector fields for momentum and position type variables on LM lift to operators on $AM \longrightarrow LM$ that provide the Dirac quantization rules that mirror that analogous results in the standard geometric quantization theory. In both cases the momentum and position type variables lead to single Hamiltonian vector fields. On the other hand we have seen above that in 4-symplectic geometry the quadratic Hamiltonian based on the spacetime metric tensor leads to the set of four Hamiltonian vector fields given in (7.3) above. It is clear that this set of Hamiltonian vector fields offers new possibilities for constructing quantum operators associated with the "free Hamiltonian" that is not available in the standard theory. Consider a spacetime manifold that admits a spin structure. In [4] it is argued that the assignment

$$X_{\hat{f}}{}^{i} r_i \longrightarrow \mathcal{P}_{\hat{g}} = -i\hbar\gamma_i X_{\hat{g}}{}^{i} \quad , \tag{7.4}$$

where the γ_i are the Dirac matrices, is a natural formulation of the Dirac operator when the $X_{\hat{g}}{}^{i}$ are lifted to the spin bundle over the the bundle of orthonormal frames. More details on how one accomplishes this lifting will be reported in a future publication [12].

There are many details that have not been addressed in this paper concerning the structure of a geometric quantization theory based on n-symplectic geometry. Among other things we have not indicated how one can lift the vector-valued Hamiltonian vector fields for arbitrary allowable observables, nor have we addressed the problem of constructing polarizations and the associated Hilbert spaces for the momentum and position type prequantization operators obtained in Section 4. We hope to return to these and other related problems in future papers.

APPENDIX

Proof of Theorem 3.1

The proof is essentially the same as the proof of Theorem 4.2.2, page 277 in [10].

For each $\xi \in \mathcal{G}$ we have $\hat{g}(\Phi_{exp(t\xi)}(u)) = \hat{g}(u)$ since $\hat{g}$ is invariant. Differentiating at t=0 we have

$$d\hat{g}(\xi_{LM}) = 0 \tag{A1}$$

which implies that

$$\mathbf{L}_{X_{\hat{J}(\xi)}}(\hat{g}) = 0 \quad . \tag{A2}$$

Since we know that $\hat{J}(\xi) = \hat{J}^i(\xi) r_i \in HF^1$, (A2) implies that

$$\{\hat{J}(\xi), \hat{g}\} = 0 \quad . \qquad (A3)$$

Hence by Lemma 3.1 we know that

$$\hat{J}^i(\xi)(F^i_t(u)) = \hat{J}^i(\xi)(u) \quad \text{for all} \quad \xi \in \mathcal{G} \quad . \qquad (A4)$$

By the definition of n-momentum mapping this implies

$$J^i(F^i_t(u)) = J^i(u) \qquad \blacksquare \qquad (A5)$$

Proof of Theorem 5.2.

We give a local coordinate proof of the theorem. Using the map ψ_α defined in (5.34) it is easy to verify the formulas

$$\psi_{\alpha *}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} \quad , \tag{A6}$$

$$\psi_{\alpha *}(\frac{\partial}{\partial \pi^a_b}) = \alpha_a \frac{\partial}{\partial p_b} \quad . \tag{A7}$$

First suppose $p \geq 2$ and fix a point $u \in LM$ and let $w = \psi_\alpha(u)$. We evaluate (5.35) at u using (A6), (A7) and the local coordinate expressions for $X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}(u)$ given in (2.14).

$$\begin{aligned} & p!\psi_{(\alpha,c)*}(X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}(u)\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{p-1}}) \\ &\qquad = \frac{p!}{(p-1)!} f^{j_1 j_2 \ldots j_{p-1} k}(x)\pi^{i_1}_{j_1}(u)\alpha_{i_1}\pi^{i_2}_{j_2}(u)\alpha_{i_2}\cdots\pi^{i_{p-1}}_{j_{p-1}}(u)\alpha_{i_{p-1}}\frac{\partial}{\partial x^k}(w) \\ &\qquad - \frac{p!}{p!}\left(\frac{\partial f^{j_1 j_2 \ldots j_p}}{\partial x^a}\pi^{i_1}_{j_1}(u)\alpha_{i_1}\pi^{i_2}_{j_2}(u)\alpha_{i_2}\cdots\pi^{i_{p-1}}_{j_{p-1}}\alpha_{i_{p-1}}\pi^b_{j_p}(u)\right)\alpha_b\frac{\partial}{\partial p_a}(w) \\ &\qquad - \frac{p!}{p!}\left(T_a^{i_1 i_2 \ldots i_{p-1} b}(u)\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{p-1}}\right)\alpha_b\frac{\partial}{\partial p_a}(w) \end{aligned} \tag{A8}$$

We note that when $p = 1$ the last term involving the arbitrary component $T_a^{i_1 i_2 \ldots i_{p-1} b}$ does not occur in this formula, and when $p \geq 2$ the last term in (A8) vanishes by (2.15) since

$$T_a^{i_1 i_2 \ldots i_{p-1} b}(u)\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{p-1}}\alpha_b = T_a^{(i_1 i_2 \ldots i_{p-1} b)}(u)\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{p-1}}\alpha_b \quad . \tag{A9}$$

Using $\pi^i_j(u)\alpha_i = p_j(w)$ back in (A8) we obtain for $p \geq 1$

$$\begin{aligned} & p!\psi_{(\alpha,c)*}(X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}}(u)\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{p-1}}) \\ &\qquad = p f^{j_1 j_2 \ldots j_{p-1} k}(x)p_{j_1}(w)p_{j_2}(w)\cdots p_{j_{p-1}}(w)\frac{\partial}{\partial x^k}(w) \\ &\qquad - \left(\frac{\partial f^{j_1 j_2 \ldots j_p}}{\partial x^a}p_{j_1}(w)p_{j_2}(w)\cdots p_{j_p}(w)\right)\frac{\partial}{\partial p_a}(w) \quad . \end{aligned} \tag{A10}$$

This vector at $w \in T^*M$ is $X_{\tilde{f}}(w)$ where $X_{\tilde{f}}$ is the Hamiltonian vector field of the function $\tilde{f}$ determined by $\hat{f}$ by (2.20).

We still need to check that we get unique tangent vectors on T^*M from the many-to-one map ψ_α. Denote right translation on LM by $h \in GL(n)$ by R_h, and let $\bar{u} = R_h(u)$ for $h \in G_\alpha$ so that $h^i_j\alpha_i = \alpha_j$ and $\psi_\alpha(\bar{u}) = \psi_\alpha(u)$. Then using (2.2) we have

$$\pi^i_j(\bar{u})\alpha_i = \pi^i_j(R_h(u))\alpha_i = (h^{-1})^i_k\pi^k_j(u)\alpha_i = \pi^k_j(u)\alpha_k \quad . \tag{A11}$$

Hence if we rewrite (A8) with u replaced everywhere by $\bar{u}$ we again obtain (A10). ∎

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