n-symplectic Hamilton-Jacobi Theory

L. K. Norris
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205

April 20, 2001

Abstract

In previous work n-symplectic geometry on the adapted frame bundle $\lambda : L_\pi E \to E$ of an $n = (m+k)$-dimensional fiber bundle $\pi : E \to M$ has been used to forumulate covariant Lagrangian field theory that is standardly formulated on the bundle $J^1\pi$ of 1 jets of sections of π. In this paper we set up an n-symplectic Hamilton-Jacobi equation in order to identify the analogue of a polarization that plays an important role in geometric quantization theory. We find that a local solution of the n-symplectic H-J equation yields a locally defined $H = GL(m) \times GL(k)$ subbundle of $L_\pi E$. This suggests that the gobal structure on $L_\pi E$ that is generated by local solutions of the Hamilton-Jacobi equations is a foliation of $L_\pi E$ by H subbundles. Such a foliation of $L_\pi E$ is shown to exist for the k-tuple of scalar fields on Minkowski spacetime.

Keywords: symplectic geometry, n-symplectic geometry, multisymplectic geometry, frame bundle, Hamiltonian field theories, Poisson bracket, jet bundles, contact structure.

MS classification: 53 C 15, 57 R 15, 58 F 05

I Introduction

The n-symplectic formulation of covariant Lagrangian field theory was developed in two recent publications. In [6] it was shown that a Lagrangian field theory on the bundle of 1-jets $J^1\pi$ of sections of $\pi : E \rightarrow M$ lifts in a natural way to an $H = GL(m) \times GL(k)$ principal bundle $\rho : L_\pi E \rightarrow J^1\pi$, where $m = dim(M)$ and $dim(E) = n = m + k$. One can then use the lifted Lagrangian to define an n-symplectic structure, the Cartan-Hamilton-Poincare n-symplectic structure. The algebra of observables defined by this n-symplectic structure was discussed in [10], together with a generalized Legendre transformation. Given this structure one can seek to generalize the Kostant-Souriau [4, 13] theory to a geometric quantization theory of fields based on the n-symplectic formalism. To do this one must generalize the concept of polarization of a symplectic manifold to the n-symplectic setting, and that is the subject of this paper.

On any bundle of linear frames LE of an n-dimensional manifold E there is a canonical R^n - valued 1-form, the soldering 1-form $\hat{\theta}$, that in n-symplectic geometry plays the role of a canonically defined generalized symplectic potential. The manifold $(LE, d\hat{\theta})$ is then an n-symplectic manifold. The reader is referred to the literature [3, 5, 6, 8, 9, 10] for the details of the resulting geometry. When E is a fiber bundle $\pi : E \rightarrow M$ over a manifold M the fiber structure on E gives rise to a reduction of LE to a certain subbundle, the bundle of adapted linear frames $L_\pi E$ [5] . The convention we use is that a frame $(p, e_1, e_2, \ldots, e_{m+k})$ is in $L_\pi E$ if the last k vectors in the frame are vertical with respect to π. The n-symplectic geometry on LE restricts to $L_\pi E$ to define an n-symplectic geometry with a restricted class of observables. In [6] it was shown that the rank 1 observables of the canonical n-symplectic geometry on $L_\pi E$ are kinematical in nature in that they represent variational vector fields.

In order to tie the geometry to the physics and to find dynamical algebras of observables McLean and Norris introduced a modified n-symplectic geometry on $L_\pi E$ based on a Lagrangian defined on $J^1\pi$. The structure of $L_\pi E$ is such that it is an $H = GL(m) \times GL(k)$ principal bundle $\rho : L_\pi E \rightarrow J^1\pi$. A Lagrangian $\mathcal{L}$ on $J^1\pi$ then lifts under the projection ρ to define a Lagrangian $L = \rho^*(\mathcal{L})$ on $L_\pi E$. McLean and Norris then defined the Cartan-Hamilton-Poincaré (CHP) 1-forms $\hat{\theta}_L$ (see equations (III.4) and (III.5) below). It is known [6] that when L is non-zero the bundle $(L_\pi E, d\hat{\theta}_L)$ is an n-symplectic manifold. In [10] a reduction method was used to identify the observables defined by this n-symplectic structure. In this paper we take another step toward a geometric quantization theory of fields and seek to identify the analogue of a polarization that plays a central in the Kostant-Souriau [4, 13] theory. Presumably the analogue of a polarization will be of fundamental importance in a generalization based on n-symplectic geometry. The problem of defining a polarization of $L_\pi E$ is non-trivial in that on $L_\pi E$ the fiber dimension and the base space dimension are not equal as is the case on the canonical symplectic manifold T^*M. To find an analogue we formulate an n-symplectic Hamilton-Jacobi equation, and by analogy with standard theory assume that a local solution of this equation will be a generating function for an n-symplectic polarization. The formulation of the n-symplectic Hamilton-Jacobi equation will involve a section $\sigma : E \to J^1\pi$, and here we will assume that we can find a global solution, leaving aside for future work the questions and implications regarding the existence of such global sections.

The structure of the paper is as follows. In section 2 we give a brief description of $L_\pi E$, including two important sets of coordinates, momentum coordinates and Lagrangian coordinates. The Lagrangian coordinates will be seen to be naturally adapted to the bundle $\rho : L_\pi E \to J^1\pi$. In section 3 we briefly review the n-symplectic struture on $L_\pi E$ defined by a Lagrangian on $J^1\pi$, and review the n-symplectic Legendre transformation from $L_\pi E$ to the momentum phase space Q_L. It will turn out that a local solution of the n-symplectic Hamilton-Jacobi equations will define a section of the bundle $Q_L \to E$. In section 4 we formulate the n-symplectic Hamilton-Jacobi equation, and show that this generalized equation contains both a Hamilton-Jacobi equation of the Cartheodory-Rund [1, 11] type together with a generalized canonical equation. Solutions of the generalized equation depend on a parameter $\tau(m)$ where m is the dimension of the base manifold M. With special choices of the parameter solutions will also be solutions of the Cartheodory-Rund [1, 11] or de Donder-Weyl [2, 15] equations.

In section 5 we examine the local solutions of the n-symplectic Hamilton-Jacobi theory and show that such solutions define H subbundles of both Q_{L} and $L_\pi E$. We propose then that the global structure on $L_\pi E$ that plays the role of an n-symplectic polarization is a foliation of $L_\pi E$ by H subbundles. The n-symplectic canonically conjugate variables on these H subbundles are $(\hat{\pi}_\alpha, \hat{S}^\beta)$, where $\hat{\pi}_\alpha$ are the momentum coordinates, and S^α are solutions of the n-symplectic Hamilton-Jacobi equations. As an example we apply the theory in section 6 to the n-tuple of scalar fields on Minkowski spacetime. Assuming a generic potential function in such a Lagrangian, the n-symplectic Hamilton-Jacobi equations are consistent if and only if the scalar fields are massless. In this trivial bundle setting we obtain a complete integral of the Hamilton-Jacobi equation, with the H-subbundles parameterized by the intial values of the field velocities. Section 7 contains a brief summary and discussion of the results, and a few facts about n-symplectic geometry are collected together in the appendix in section 8.

II The Canonical n-symplectic structure on $L_\pi E$

On LE there exists the canonically defined $\mathbb{R}^{m+k}$-valued soldering 1-form $\hat{\theta} = \theta^\alpha \hat{r}_\alpha$, where $(\hat{r}_\alpha)$ denotes the standard basis of $\mathbb{R}^{m+k}$. If X is a tangent vector to LE at $u = (e, e_\alpha)$ then

$$\hat{\theta}_u(X) = e^\alpha(\lambda_*(X))\hat{r}_\alpha \tag{II.1}$$

where (e^α) denotes the coframe dual to the frame (e_α) and $\lambda : LE \to E$ is the projection.

To see the significance of the soldering 1-forms $(\theta^\alpha) = (\theta^i, \theta^A)$ in the field theory setting we introduce canonical coordinates. Let $(z^\alpha) = (x^i, y^A)$ denote local coordinates on E, where (x^i) are local coordinates on M, and (y^A) are fiber coordinates on E. Then these coordinates define local canonical coordinates [6] $(z^\alpha, \pi^\beta_\mu)$ on LE by the formulas

$$z^\alpha(u) = z^\alpha \circ \lambda(u) \ , \qquad \pi^\mu_\nu(u) = e^\mu(\frac{\partial}{\partial z^\nu}) \ , \ \forall \ u \in LE$$

where we are thus using the same notation for the horizontal coordinates as we use for the base space coordinates. In these coordinates the soldering 1-forms have the canonical form

$$\theta^\alpha = \pi^\alpha_\beta dz^\beta$$

Upon restricting the soldering 1-forms to the subbundle $L_\pi E$ and switching to *Lagrangian coordinates* $(x^i, y^A, u^i_j, u^A_B, u^A_j)$ (see the appendix) we find that the (θ^i, θ^A) take the special forms

$$\theta^i = u^i_j dx^j \tag{II.2}$$

$$\theta^A = u^A_B(dy^B - u^B_k dx^k) \tag{II.3}$$

where (u^i_j) and (u^A_B) are non-singular matrix-valued functions on $L_\pi E$, and $u^B_k = \rho^*(y^B_k)$ with y^B_k the local velocity coordinates on $J^1\pi$. These formulas show that the contact structure (see references ([12]) and [6]) of $J^1\pi$ is *unified* in the soldering 1-form $\hat{\theta}$ on $L_\pi E$.

In the n-symplectic theory the vector-valued 1-form $\hat{\theta} := \theta^\alpha \otimes \hat{r}_\alpha$ is considered as a generalization of the canonical symplectic potential $p_i dq^i$ on T^*E. The closed vector-valued 2-form $d\hat{\theta}$ is also nondegenerate in the sense that $X \,\lrcorner\, d\hat{\theta} = 0 \Leftrightarrow X = 0$. These ideas led to the following definition [8] :

Definition II.1 *The pair $(LE, d\hat{\theta})$ is an n-symplectic manifold.*

The observables associated with this canonical n-symplectic structure on LE split up into two disjoint sets, namely a Poisson algebra of symmetric $\otimes^p \mathbb{R}^n$-valued functions and a graded Poisson algebra of anti-symmetric $\otimes^p \mathbb{R}^n$-valued functions. In the symmetric case the observables are, in local canonical coordinates, polynomials in the n-symplectic momentum coordinates with coefficients that are constant on the fibers of LE. In the anti-symmetric case the observables are, in local canonical coordinates, Grassman polynomials in the momenta. The homogeneous polynomials in both cases correspond to contravariant tensor fields on M, and the n-symplectic Poisson brackets when restricted to this subset of n-symplectic observables are known [9] to be the frame bundle version of the Schouten-Nijenhuis brackets [14, 7]. The reader is referred to the literature [9] for the details. However, the n-symplectic bracket contains more than the Schouten-Nijenhuis bracket, and this difference and its implications are discussed in section 7.

III The Modified n-symplectic Structure Defined by a Lagrangian L

In reference [6] it is shown that $L_\pi E$ is a principal $H = GL(m) \times GL(k)$ bundle over the bundle $J^1\pi$ of 1-jets of sections of π. Letting $\rho : L_\pi E \to J^1\pi$ denote the projection, one can define the CHP 1-forms $\theta^\alpha_{\mathrm{L}}$ on $L_\pi E$ by the formulas

$$\theta^i_{\mathrm{L}} := \tau \mathrm{L} \theta^i + E^{*i}_A(\mathrm{L})\theta^A \tag{III.4}$$

$$\theta^A_{\mathrm{L}} := \theta^A \tag{III.5}$$

where $\tau = \tau(m)$ is a positive constant depending on the dimension m of the base manifold M, and E^{*i}_A denotes the fundamental vertical vector field on $L_\pi E$ corresponding to the element E^i_A in the standard basis (E^α_β) of $gl(n)$. Local coordinate formulas for these fundamental vertical vector fields are given in the appendix. For different values of τ one can obtain the de Donder-Weyl theory [2, 15] and the Caratheodory-Rund theory [1, 11] as special cases of the formalism presented in reference [6]. MacLean and Norris [6] also proved the one may easily construct the CHP m-form on $J^1\pi$ from the CHP 1-forms on $L_\pi E$.

One can define this modified soldering 1-form using a frame bundle version of the Legendre transformation [6]. Given a Lagrangian $\mathrm{L} : L_\pi E \to \mathbb{R}$ we obtain a mapping $\phi_{\mathrm{L}} : L_\pi E \to LE$ given by

$$\phi_{\mathrm{L}}(u) = \phi_{\mathrm{L}}(e, e_i, e_A) = \left(e, \frac{1}{\tau \mathrm{L}(u)} e_i, e_A - \frac{1}{\tau \mathrm{L}(u)} E^{*a}_A(\mathrm{L})(u) e_a\right) \tag{III.6}$$

This mapping is well-defined provided the Lagrangian is non-zero, and for the rest of this paper we will assume this condition. This mapping is as the *n-symplectic Legendre transformation*. In reference [10] it is shown that $\hat{\theta}_L$ on $L_\pi E$ and $\hat{\theta}$ on Q_{L} are related as one might expect by the formula

$$\hat{\theta}_{\mathrm{L}} = \phi^*_{\mathrm{L}}(\hat{\theta}) \tag{III.7}$$

We note that the CHP 1-forms given in (III.4) and (III.5) above can be expressed, using the Lagrangian coordinates defined in the appendix, in the form

$$\theta^i = u^i_j(-\mathcal{H}^j_k dx^k + p^j_B dy^B) \tag{III.8}$$

$$\theta^A = u^A_B(-u^A_k dx^k + dy^B) \tag{III.9}$$

where we have introduced the definitions

$$\mathcal{H}^i_j := p^i_A u^A_j - \tau \mathrm{L}\delta^i_j \ , \qquad p^i_A := \frac{\partial \mathrm{L}}{\partial u^A_i} \tag{III.10}$$

Using the additional definitions

$$\left(h^\alpha_\beta\right) = \begin{pmatrix} -\mathcal{H}^k_j & p^k_A \\ -u^E_j & \delta^E_A \end{pmatrix} , \qquad \left((\Delta u)^\alpha_\beta\right) = \begin{pmatrix} u^k_j & 0 \\ 0 & u^E_A \end{pmatrix} \tag{III.11}$$

equations (III.8) and (III.9) can be written in the following compact form:

$$\theta^\alpha_{\mathrm{L}} = ((\Delta u)^\alpha_\beta) h^\beta_\gamma dz^\gamma \tag{III.12}$$

For later reference we note here that when $\mathrm{L} \neq 0$ the matrix (h^α_β) has an inverse given by

$$\left((h^{-1})^\alpha_\beta\right) = \frac{1}{\tau \mathrm{L}} \begin{pmatrix} \delta^k_j & -p^k_A \\ u^E_j & -(h^{-1})^E_A \end{pmatrix} , \qquad (h^{-1})^E_A = p^a_A u^E_a - \tau \mathrm{L}\delta^E_A \tag{III.13}$$

The matrix $((\Delta u)^\alpha_\beta)$ in (III.12) transforms [6] under the group H of the bundle $\rho : L_\pi E \to J^1\pi$ while the second factor (h^α_β) is H-invariant. We will use these facts in the next section to set up a generalized Hamilton-Jacobi equation that will bring the modified n-symplectic potential to canonical form.

In [10] it was shown that if the lifted Lagrangian is non-zero, then $(L_\pi E, d\hat{\theta}_{\mathrm{L}})$ is an n-symplectic manifold, and that ϕ_{L} is a diffeomorphism onto its image $Q_{\mathrm{L}} \subset LE$. Hence Q_{L} is a natural candidate for the momentum phase space of fields.

IV The n-symplectic Hamilton-Jacobi Equation

In symplectic geometry a local solution of the Hamilton-Jacobi equation on T^*M plays the role of a generating function for a polarization of T^*M [16] . Here we seek an n-symplectic analogue of such a polarization. The problem of defining a polarization of $L_\pi E$ is non-trivial in that on $L_\pi E$ the fiber dimension and the base space dimension are not equal as is the case on T^*M. To find the form of n-symplectic polarizations we formulate an n-symplectic Hamilton-Jacobi type equation. Local solutions of this equation should then generate the polarizations of $L_\pi E$ that we are seeking.

We seek sections $\sigma : E \rightarrow J^1\pi$ that satisfy the equations

$$(\sigma^* h^\alpha_\beta) = \left(\frac{\partial S^\alpha}{\partial z^\beta}\right) \tag{IV.14}$$

for some $m+k$ functions S^α defined on open subsets of E. This set of equations is the H-invariant form of the generalized Hamilton-Jacobi equations proposed in [6].

For convenience we will denote with an over-tilde objects on $J^1\pi$ pulled back to E using σ. Thus, for example, $\tilde{\mathcal{H}}^i_j = \mathcal{H}^i_j \circ \sigma$ and $\tilde{p}^i_A = p^i_A \circ \sigma$. Then we get from (III.11) and (IV.14) the equations

$$\textbf{(a)}\ \ \tilde{\mathcal{H}}^i_j = -\frac{\partial S^i}{\partial x^j}\ , \qquad \textbf{(b)}\ \ \tilde{p}^i_A = \frac{\partial S^i}{\partial y^A} \tag{IV.15}$$

$$\textbf{(a)}\ \ \tilde{u}^B_j = -\frac{\partial S^A}{\partial x^j}\ , \qquad \textbf{(b)}\ \ \tilde{\delta}^A_B = \frac{\partial S^A}{\partial y^B} \tag{IV.16}$$

Notice that equation (IV.14) makes sense because h^α_β is H invariant and passes to the quotient $J^1\pi = L_\pi E/H$. We will shortly use such a section to define H subbundles of $L_\pi E$.

Consider first equations (IV.16), the second of which implies that

$$S^A = y^A - \psi^A(x^a) \tag{IV.17}$$

in terms of a new set of functions $\psi^A(x^i)$. Substitution of this result into the first of equations (IV.16) yields

$$\tilde{u}^B_j = \frac{\partial \psi^A}{\partial x^j} \tag{IV.18}$$

so that the section $\sigma : E \rightarrow J^1\pi$ is holonomic and determines a section $p \rightarrow (x^i(p), \psi^A(p))$ of $\pi : E \rightarrow M$.

Turning next to equations (IV.15) we note that $\mathcal{H}^i_j = p^i_B u^B_j - \tau \mathrm{L} \delta^i_j$ and $p^i_A = \frac{\partial \mathrm{L}}{\partial u^A_i}$ are functions of the coordinates x^i, y^A, and u^A_i, so that $\tilde{\mathcal{H}}^i_j$ can be considered as a function of x^i and y^A. Hence equations (IV.15) can be combined into the single equation

$$\tilde{\mathcal{H}}^i_j\left(x^a, y^B, \tilde{p}^i_A = \frac{\partial S^i}{\partial y^A}\right) = -\frac{\partial S^i}{\partial x^j} \tag{IV.19}$$

This equation is clearly a generalized Hamiton-Jacobi equation and is similar to the Hamilton-Jacobi equations in the Caratheodory-Rund [1, 11] and de Donder-Weyl [2, 15] canonical theories.

Theorem IV.1 *Suppose that the section $\sigma : E \to J^1\pi$ satisfies equation (IV.14). Then σ defines on $U \subset M$ a section $\psi : U \to E$. Moreover the locally defined functions (S^i, S^A) are such that:*

1. *The functions S^A are given by $S^A = y^A - \psi^A(x^a)$*
2. *The functions S^i and ψ^A satisfy*
   (a) *the generalized Hamilton-Jacobi equation $\tilde{\mathcal{H}}^i_j(x^a, y^E, u^F_b = \frac{\partial \psi^F}{\partial x^b}, p^c_D = \frac{\partial S^c}{\partial y^D}) = -\frac{\partial S^i}{\partial x^j}$*
   (b) *the generalized canonical equations $\left(\frac{\partial \tilde{\mathcal{H}}^i_j}{\partial y^A}\right) = -\left(\frac{\partial \tilde{p}^i_A}{\partial x^j}\right)$.*

Proof

The integrability conditions of equation (IV.14) are

$$\left(\frac{\partial \tilde{h}^\alpha_\beta}{\partial z^\mu}\right) - \left(\frac{\partial \tilde{h}^\alpha_\mu}{\partial z^\beta}\right) = 0 \tag{IV.20}$$

This equation splits up into six sets of equations:

$$0 = \left(\frac{\partial \tilde{\mathcal{H}}^i_j}{\partial x^k}\right) - \left(\frac{\partial \tilde{\mathcal{H}}^i_k}{\partial x^j}\right) \tag{IV.21}$$

$$0 = -\left(\frac{\partial \tilde{\mathcal{H}}^i_j}{\partial y^A}\right) - \left(\frac{\partial \tilde{p}^i_A}{\partial x^j}\right) \tag{IV.22}$$

$$0 = \left(\frac{\partial \tilde{p}^i_A}{\partial y^B}\right) - \left(\frac{\partial \tilde{p}^i_B}{\partial y^A}\right) \tag{IV.23}$$

$$0 = \left(\frac{\partial \tilde{h}^A_j}{\partial x^k}\right) - \left(\frac{\partial \tilde{h}^A_k}{\partial x^j}\right) \tag{IV.24}$$

$$0 = \left(\frac{\partial \tilde{h}^A_j}{\partial y^B}\right) - \left(\frac{\partial \tilde{h}^A_B}{\partial x^j}\right) \tag{IV.25}$$

$$0 = \left(\frac{\partial \tilde{h}^A_C}{\partial y^B}\right) - \left(\frac{\partial \tilde{h}^A_B}{\partial y^C}\right) \tag{IV.26}$$

Equation (IV.26) is identically satisfied, while equations (IV.21) and (IV.23) - (IV.25) imply equations (IV.17) - (IV.19), which yields part 1 and part 2-a of the theorem. Equation (IV.22) is the generalized canonical equation, which is part 2-b of the theorem. ■

Remark: We note that every solution of the the generalized canonical equations (2b) in the theorem will also be a solution of the de Donder - Weyl equations [11]

$$\frac{\partial H}{\partial y^A} = -\frac{\partial p^i_A}{\partial x^i} \tag{IV.27}$$

when we put $\tau = \frac{1}{m}$. This is because this equation is the trace of the uncontracted n-symplectic equation

$$\left(\frac{\partial \tilde{\mathcal{H}}^i_j}{\partial y^A}\right) = -\left(\frac{\partial \tilde{p}^i_A}{\partial x^j}\right) \tag{IV.28}$$

that appears in the above theorem, since the de Donder-Weyl Hamiltonian H is identically equal to the trace $\tilde{H}^i_i$ of the $\tilde{H}^i_j$ when $\tau = \frac{1}{m}$. It is clear that the uncontracted equation is more restrictive than the contracted equation. Solutions (S^i, S^A) serve to define the canonical field variables $(\hat{S}^\alpha, \hat{\pi}_\beta)$ on $L_\pi E$. All other observables will then be polynomials in the $\hat{\pi}_\alpha$ with coefficients that are functions of the variables S^α. For this reason we shall refer to the canonical variables $(\hat{S}^\alpha, \hat{\pi}_\beta)$ as basic states of the field. In the example we will examine the basic states of the n-tuple of scalar fields on Minkowski spacetime.

V Canonical variables $\hat{\pi}_\alpha$ and $\hat{S}^\beta$

In this paper we pursue the local theory and will consider the global theory in future work. Accordingly let us assume that we have found a global section $\sigma : E \to J^1\pi$ together with functions (S^i, S^A) defined locally on $U \subset E$ and satisfying the above equations. For such a section we define a subset $B_\sigma \subset L_\pi E$ by the equation

$$B_\sigma := \rho^{-1}(\sigma(E))$$

B_σ is clearly an H bundle over the section $\sigma(E)$, with projection mapping $\rho|_{B_\sigma}$. By composition with the projection $\pi^{1,0} : J^1\pi \to E$ we may think of B_σ as an H bundle over $U \subset E$.

It is not difficult to show that in fact the local solutions (S^α) of the Hamilton-Jacobi equation defines a local section of Q_{L}. In fact if we denote this section by $\sigma_1 : E \to Q_{\mathrm{L}}$, then it is not difficult to show from the definition of ϕ_L that

$$(\phi_L)_*(\frac{\partial}{\partial x^i} + \tilde{u}_i^A \frac{\partial}{\partial y^A}) = \frac{\partial}{\partial S^i}$$

It then follows that $B_\sigma = (\phi_L)^{-1}(\sigma_1(E)) \cdot H$. We then let $\mu := (\phi_L)^{-1}(\sigma_1(E))$.

On $L_\pi E$ the CHP 1-forms θ^i_{L} are given by

$$\theta^i_{\mathrm{L}} = -\pi^i_j \mathcal{H}^j_k dx^k + \pi^i_j p^j_A dy^A = \pi^i_j(-\mathcal{H}^j_k dx^k + p^j_A dy^A) \tag{V.29}$$

If we evaluate these 1-forms on the section μ we find for $u \in \mu(E)$

$$\theta^i_{\mathrm{L}}(u) = \tilde{\pi}^i_j dS^j \tag{V.30}$$

and

$$\theta^A_{\mathrm{L}}(u) = \tilde{\pi}^A_B dS^B \tag{V.31}$$

Then for each $h \in H$ we have

$$\theta^i_{\mathrm{L}}(u \cdot h) = (h^{-1})^i_k \theta^k_{\mathrm{L}}(u) = (h^{-1})^i_k \tilde{\pi}^k_j(u) dS^j = \pi^i_j(u) dS^j \tag{V.32}$$

since both θ^i_{L} and π^i_j transform tensorially under the group H and $\pi^i_j(u \cdot h) = (h^{-1})^i_k \pi^k_j(u)$. Similarly for θ^A_{L}. Hence the restriction $\hat{\theta}_{\mathrm{L}}|_{B_\sigma}$ of the modified n-symplectic potential $\hat{\theta}_{\mathrm{L}}$ is in the canonical form $(\theta^\alpha_{\mathrm{L}}|_{B_\sigma}) = (\pi^i_j dS^j, \pi^A_B dS^B)$ with respect to the coordinates $(S^\alpha, \pi^i_j, \pi^A_B)$ on B_σ. On B_σ we therefore have the following 2-forms:

$$\begin{aligned} d\theta^i_{\mathrm{L}}|_{B_\sigma} &= d\pi^i_j \wedge dS^j && \text{(V.33)} \\ d\theta^A_{\mathrm{L}}|_{B_\sigma} &= d\pi^A_B \wedge dS^B && \text{(V.34)} \end{aligned}$$

$d\hat{\theta}_{\mathrm{L}}|_{B_\sigma}$ is clearly closed and non-degenerate, so that $(B_\sigma, d\hat{\theta}_{\mathrm{L}}|_{B_\sigma})$ *is an n-symplectic manifold.* This implies that $\hat{\pi}_\alpha = (\pi^j_i \hat{r}_j, \pi^B_A \hat{r}_B)$ and $\hat{S}^\beta = (S^i \hat{r}_i, S^A \hat{r}_A)$ are canonically conjuate pairs of n-symplectic observables with respect to the modified n-symplectic form $d\hat{\theta}_{\mathrm{L}}|_{B_\sigma}$ on B_σ. In addition we know that the n-symplectic Hamiltonian vector field $X_{\hat{\pi}_i}$ determined by the momentum coordinate $\hat{\pi}_i = \pi^j_i \hat{r}_j$ is $X_{\hat{\pi}_i} = \frac{\partial}{\partial S^i}$. If the integrability conditions of equations

n-symplectic canonical variables on B_σ	
Conjugate variables	**Hamiltonian vector fields**
$\hat{S}^i = S^i \hat{r}_i$ (no sum on i)	$X_{\hat{S}^i} = -\frac{\partial}{\partial \pi^i_i}$ (no sum on i)
$\hat{\pi}_i = \pi^j_i \hat{r}_j$	$X_{\hat{\pi}_i} = \frac{\partial}{\partial S^i} = \frac{1}{\tau \tilde{\mathrm{L}}}\left(\frac{\partial}{\partial x^i} + \tilde{u}^A_i \frac{\partial}{\partial y^A}\right)$
$\hat{S}^A = S^A \hat{r}_A$ (no sum on A)	$X_{\hat{S}^A} = -\frac{\partial}{\partial \pi^A_A}$ (no sum on A)
$\hat{\pi}_A = \pi^B_A \hat{r}_B$	$X_{\hat{\pi}_A} = \frac{\partial}{\partial S^A} = \frac{1}{\tau \tilde{\mathrm{L}}}\left(-\tilde{p}^i_A \frac{\partial}{\partial x^i} - (\tilde{h}^{-1})^B_A \frac{\partial}{\partial y^B}\right) \equiv \frac{\partial}{\partial y^A} - \frac{1}{\tau \tilde{\mathrm{L}}}\tilde{p}^i_A \frac{d}{dx^i}$

Table 1: n-symplectic canonical variables

(IV.14) are satisfied then the functions (S^α) define a new coordinate system on E, and we have

$$\frac{\partial}{\partial S^i} = \frac{\partial x^k}{\partial S^i}\frac{\partial}{\partial x^k} + \frac{\partial y^B}{\partial S^i}\frac{\partial}{\partial y^B} \tag{V.35}$$

Using (III.13) for the matrix ($\frac{\partial z^\alpha}{\partial S^\beta}$) we find

$$\frac{\partial}{\partial S^i} = \frac{1}{\tau \tilde{\mathrm{L}}}\left(\frac{\partial}{\partial x^i} + \tilde{u}^A_i \frac{\partial}{\partial y^A}\right) \tag{V.36}$$

Similarly we find

$$\frac{\partial}{\partial S^A} = \frac{\partial}{\partial y^A} - \frac{1}{\tau \tilde{\mathrm{L}}}\tilde{p}^i_A \frac{d}{dx^i} \tag{V.37}$$

The conjugate variables and their n-symplectic Hamiltonian vector fields are displayed in Table 1.

The canonical commutation relations of the canonical variables ($\hat{\pi}_\alpha,\ \hat{S}^\alpha$) on B_σ are [8]

$$\{\hat{S}^\alpha, \hat{S}^\beta\} = 0\ , \qquad \{\hat{\pi}_\alpha, \hat{\pi}_\beta\} = 0\ , \qquad \{\hat{\pi}_\alpha, \hat{S}^\beta\} = \delta^\beta_\alpha \hat{r}_\beta \tag{V.38}$$

Inspection of these relations revels that they divide into three sets:

$$\{\hat{S}^i, \hat{S}^j\} \;=\; 0\ , \quad \{\hat{\pi}_i, \hat{\pi}_j\} = 0\ , \quad \{\hat{\pi}_i, \hat{S}^j\} = \delta^j_i \hat{r}_j \tag{V.39}$$

$$\{\hat{S}^A, \hat{S}^B\} \;=\; 0\ , \quad \{\hat{\pi}_A, \hat{\pi}_B\} = 0\ , \quad \{\hat{\pi}_A, \hat{S}^B\} = \delta^B_A \hat{r}_B \tag{V.40}$$

$$\{\hat{S}^i, \hat{S}^A\} \;=\; 0\ , \quad \{\hat{\pi}_i, \hat{\pi}_A\} = 0\ , \quad \{\hat{\pi}_A, \hat{S}^i\} = 0\ , \quad \{\hat{\pi}_i, \hat{S}^A\} = 0 \tag{V.41}$$

Equations (V.39) are the m-symplectic equations for $L(\mathbb{R}^m)$, while equations (V.40) are the k-symplectic equations for $L(\mathbb{R}^k)$. Finally equations (V.41) show that the full algebra on B_σ is a direct sum of these two algebras.

Theorem V.1 *The n-symplectic algebra on B_σ defined in (V.38) is the direct sum of the n-symplectic algebra on $L(\mathbb{R}^m)$ with the k-symplectic algebra on $L(\mathbb{R}^k)$. Symbolically we write*

$$< \ S^\alpha, \pi_\beta \ > = < \ S^i, \pi_j \ > \oplus < \ S^A, \pi_B \ > \tag{V.42}$$

So far we have only considered the geometry associated with a single solution of the n-symplectic Hamilton-Jacobi equation. If we can find what one would want to call a *complete integral* of these equations, that is to say solutions that are parameterized by some initial values of the fields, then we would have an H-subbundle of $L_\pi E$ for each initial condition. In at least the simpliest models we would then obtain a foliation of $L_\pi E$ by H-subbundles, and we take this foliation as the basic model of an n-symplectic polarization. In the next sedtion we find this polarization for the simple case of an n-tuple of scalar fields on Minkowski spacetime.

VI Example: The k-tuple of scalar fields on Minkowski spacetime

In this example we use the following strategy to solve the Hamilton-Jacobi-canonical equations (IV.14). We first solve the generalized canonical equation (IV.28) for a section of $\pi : E \to M$, and then substitute the section into equations (IV.15) and (IV.16) and solve for the coordinate functions (S^i, S^A). M denotes 4-dimensional Minkowski spacetime.

For the k-tuple of scalar fields we consider E as a vector bundle over M with standard fiber the linear space $\mathbb{R}^k$. We choose the general form of the Lagrangian for such a field to be

$$\mathcal{L} = \frac{1}{2}\eta^{ij}\delta_{AB}u_i^A u_j^B - \mu^2 V(y^A) + L_0 \tag{VI.43}$$

where η^{ij} are the components of the Minkowski metric tensor in inertial coordinates (x^i), and δ^{AB} are the components of the internal Euclidean metric in the fiber coordinates (y^A).

The term $V(y^A)$ denotes a general potential function that depends only on the field coordinates y^A, and L_0 is a positive constant added on to guarantee that the total Lagrangian is positive. We consider this Lagrangian as defined on $L_\pi E$ in the Lagrangian coordinates $(x^i, y^A, u^i_j, u^A_B, u^A_i)$, and rename it L. Assuming that the Hamilton-Jacobi equations are satisfied we have the following expressions that derive from the Lagrangian:

$$\begin{aligned} \tilde{u}^A_i &= \frac{\partial \psi^A}{\partial x^i} && \text{(VI.44)} \\ \tilde{p}^i_A &= \partial^i \psi_A && \text{(VI.45)} \\ \tilde{\mathcal{H}}^i_j &= (\partial^i \psi_A)(\partial_j \psi^A) - \tau \tilde{\mathrm{L}} \delta^i_j && \text{(VI.46)} \end{aligned}$$

The generalized canonical equations (IV.28) for this case are

$$\eta^{ik} \partial_j \partial_k \psi^A = \frac{\mu^2}{\tau} y^A \delta^i_j \tag{VI.47}$$

Since there is no y^A dependence on the left hand side of this equation both sides must vanish separately. The vanishing of the right hand side yields $\mu = 0$ so only massless scalar fields are compatible with the n-symplectic Hamilton-Jacobi equation. The vanishing of the left hand side of the equation implies $\partial_i \partial_j \psi^A = 0$. Hence $\psi^A = \xi^A_i x^i + k^A$ where $(\xi^A_i) \in \mathbb{R}^{m\times k}$ and $(k^A) \in \mathbb{R}^k$ are constants.

As discussed above we know that the Hamilton-Jacobi functions S^A are given by $S^A = y^A - \psi^A$. We now seek the functions S^i, which are defined in equations (IV.15). Since $\partial_i \psi^A$ are all constant , both $\tilde{\mathcal{H}}^i_j$ and $\tilde{p}^i_A$ are also constants. Hence equations (IV.15) imply

$$S^i = -\tilde{\mathcal{H}}^i_j x^j + \tilde{p}^i_A y^A + c^i \tag{VI.48}$$

where $\tilde{\mathcal{H}}^i_j = \xi^A_j \xi^i_A - \tau(\frac{1}{2}(\xi^2) + L_0)\delta^i_j$ and $\tilde{p}^i_A = \xi^i_A$. Notice that we have found a "complete integral" of the generalized Hamilton-Jacobi equation (IV.14) with the functions (S^α) depending on the parameters ξ^A_a.

Let $\Sigma_0 \subset \mathbb{R}^{m\times k}$ be the open subset of $\mathbb{R}^{m\times k}$ defined by the non-zero condition $\mathrm{L}_\xi \neq 0$. For each $\xi \in \Sigma_0$ we obtain a corresponding section $\sigma_\xi : E \to J^1\pi$ and an H-subbundle $B_{\sigma_\xi} \subset L_\pi E$, and we set

$$B_{\Sigma_0} := \cup_{\xi \in \Sigma_0} B_{\sigma_\xi} \tag{VI.49}$$

We thus have a decomposition of the portion of $L_\pi E$ on which the Lagrangian is non-zero into a disjoint union of H subbundles over E.

VII Conclusions

In this paper we have identified, in the context of the n-symplectic theory, the analogue of a polarization of a symplectic manifold. To do this we used the bundle $\rho : L_\pi E \to J^1\pi$ to lift a Lagrangian on $J^1\pi$ to a Lagrangian on $L_\pi E$. We then could exploit the n-symplectic structure defined by this Lagrangian. In particular we formulated an n-symplectic Hamilton-Jacobi equation, and found that it contained both a classical Hamilton-Jacobi type equation together with a generalized canonical equation. Local solutions of this equation were shown to define H subbundles of $L_\pi E$, and hence led us to propose a foliation of $L_\pi E$ by H subbundles as the choice of a real polarization in n-symplectic theory. The theory was applied to the n-tuple of scalar fields on Minkowski spacetime. We found that the n-symplectic Hamilton-Jacobi equation describes only massless scalar fields. In the trivial bundle setting we were able to identify a complete integral for the Hamilton-Jacobi equation, which led to a decomposition of the portion of $L_\pi E$ on which the Lagrangian is non-zero into a disjoint union of H subbundles over E.

A few remarks are in order concerning the relationship of the n-symplectic brackets to the brackets of Schouten [14] and Nijenhuis [7]. The full n-symplectic bracket is an extension of the Schouten-Nijenhuis bracket [8, 9], as can be seen as follows. For $p = 1$ the allowable Hamiltonian functions on LE decompose as [8]

$$HF^1 = T^1(LE) \oplus C^\infty(E, \mathbb{R}^n)$$

where the second factor denotes functions on LE that are constant on fibers and hence are lifts of $\mathbb{R}^n$-valued functions on E. The first factor $T^1(LE)$ denotes the set of $\mathbb{R}^n$-valued "tensorial" functions that transform under the standard representation of $GL(n)$ on $\mathbb{R}^n$. Such tensorial functions are in 1-1 correspondence with vector fields on E. If we write $\hat{f} \sim \vec{f} \oplus \xi$ and $\hat{g} \sim \vec{g} \oplus \eta$ for two such elements of $T^1(LE) \oplus C^\infty(E, \mathbb{R}^n)$, then the n-symplectic bracket of $\hat{f}$ and $\hat{g}$ can be expressed as

$$\{\hat{f}, \hat{g}\} \sim [\vec{f}, \vec{g}] \oplus \left(\vec{f}(\xi) - \vec{g}(\eta)\right)$$

The term in square brackets on the right hand side is the Lie bracket of vector fields $\vec{f}$ and $\vec{g}$ on E, and the piece $\vec{f}(\xi) - \vec{g}(\eta)$ is the new piece that is not part of the definition of the

Schouten-Nijenhuis [14, 7] bracket. One might think that it would be sufficient to work with the "tensorial" part of HF^1, but this is not correct, as the "coordinate observables" live in the $C^\infty(E,\mathbb{R}^n)$ part of HF^1. If one hopes to write down canonical commutation relations that include the brackets of momenta and coordinates as is done in standard theory, then one must include the extra factors in the bracket. More explicitly, suppose one has coordinates $(z^\alpha, \pi^\beta_\mu)$ on LE. Then the n-symplectic momentum and coordinate variables are, respectively,

$$\hat{\pi}_\alpha = \pi^\beta_\alpha \hat{r}_\beta \in T^1(LE) \ , \qquad \hat{z}^\lambda = z^\lambda \hat{r}_\lambda \in C^\infty(E,\mathbb{R}^n) \ \ (\text{no sum on } \lambda)$$

The n-symplectic brackets of these variables are [8]:

$$\{\hat{\pi}_\alpha, \hat{\pi}_\beta\} = \{\hat{z}^\mu, \hat{z}^\nu\} = 0 \ , \qquad \{\hat{\pi}_\alpha, \hat{z}^\beta\} = \delta^\beta_\alpha \hat{r}_\beta$$

and these brackets have the same general form as the canonical commutation (CC) relations on, say, the cotangent bundle T^*E. If one insists on using only the tensorial part of HF^1 then one will not be able to obtain these commutation relations. One might try to put the coordinate variables z^α into the tensorial part using the locally defined "radius vector" $\vec{R} = z^\alpha \frac{\partial}{\partial z^\alpha}$ which corresponds to the tensorial piece $\hat{R} = z^\alpha \pi^\beta_\alpha \hat{r}_\beta \in T^1(LE)$, but as easily checked these variables will not yield the correct CC-relations. Thus the n-symplectic brackets on LE generalize in an important way the brackets for symmetric and antisymmetric contravariant tensor fields discovered by Schouten and Nijenhuis.

VIII Appendix: The Vertically Adapted Linear Frame Bundle $L_\pi E$

Let $\pi : E \to M$ be a fiber bundle where M is m-dimensional and E is $n = m+k$-dimensional. Lower case latin indices are assumed to range over $1\ldots m$, upper case latin indices over $m+1\ldots m+k$, and greek indices over $1\ldots m+k$. This convention will be used throughout the paper.

An adapted frame at $e \in E$ is a frame where the last k basis vectors are vertical. Note that coordinate frames that come from adapted coordinates are adapted frames. The adapted frame bundle of π, denoted $L_\pi E$, consists of all adapted frames for E.

$$L_\pi E = \{(e, \{e_i, e_A\}) : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } d_u\pi(e_A) = 0\}$$

The canonical projection, $\lambda : L_\pi E \to E$, is defined by $\lambda(e, \{e_i, e_A\}) = e$.

$L_\pi E$ is a reduced subbundle of LE, the frame bundle of E (Lawson [5]). As such it is a principal fiber bundle over E. Its structure group is $\mathrm{G_v}$, the nonsingular block lower triangular matrices.

$$\mathrm{G_v} = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in \mathrm{GL}(m), B \in \mathrm{GL}(k), C \in \mathbb{R}^{km} \right\}$$

$\mathrm{G_v}$ acts on $L_\pi E$ on the right by

$$(e, \{e_i, e_A\}) \cdot \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \{(e, \{e_i A^i_j + e_A C^A_j, e_A B^A_B\})$$

If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $L_\pi E$. The *coframe* or *n-symplectic momentum* coordinates $(x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)$ on $\lambda^{-1}(U)$ are defined by, for $u = (e, \{e_i, e_A\}) \in L_\pi E$,

$$x^i(u) = x^i(e) \qquad \pi^i_j(u) = e^i(\frac{\partial}{\partial x^j}) \qquad \pi^A_B(u) = e^A(\frac{\partial}{\partial y^B})$$
$$y^A(u) = y^A(e) \qquad \pi^A_j(u) = e^A(\frac{\partial}{\partial x^j})$$

Here (e^i, e^A) is the dual frame to (e_i, e_A). We have as is customary retained the same symbols for the induced horizontal coordinates. The symbol that is missing from the above list, namely π^i_A, defined in te obvious way, vanishes identically on $L_\pi E$.

The *frame* or *n-symplectic velocity* coordinates $(x^i, y^A, v^i_j, v^A_j, v^A_B)$ on $\lambda^{-1}(U)$ are defined by, for $u = (e, \{e_i, e_A\}) \in L_\pi E$,

$$x^i(u) = x^i(e) \qquad v^i_j(u) = e_j(x^i) \qquad v^A_B(u) = e_B(y^A)$$
$$y^A(u) = y^A(e) \qquad v^A_j(u) = e_j(y^A)$$

The v coordinates, viewed together as a block triangular matrix, form the inverse of the π coordinates above. The blocks have the following relations:

$$v^i_j \pi^j_k = \delta^i_k \qquad v^A_j \pi^j_k + v^A_B \pi^B_k = 0 \qquad v^A_B \pi^B_C = \delta^A_C$$

One may construct from the previous two coordinate systems a third important system. Define $(x^i, y^A, u^i_j, u^A_j, u^A_B)$ on $\lambda^{-1}(U)$ by

$$
\begin{aligned}
x^i(u) &= x^i(e) & u^i_j &= \pi^i_j & u^A_j &= v^A_i \pi^i_j = -v^A_B \pi^B_j \\
y^A(u) &= y^A(e) & u^A_B &= \pi^A_B & &
\end{aligned}
$$

It is shown in reference [6] that the u^A_j coordinates are pull-ups of the standard jet coordinates on $J^1\pi$. As such, we will refer to these coordinates as *Lagrangian* coordinates.

We need the following formulas for the fundamental vertical vector fields $E^{*\alpha}_\beta$ on $L_\pi E$ in Lagrangian coordinates.

$$
E^{*i}_j = -u^i_k \frac{\partial}{\partial u^j_k} \qquad E^{*A}_B = -u^A_C \frac{\partial}{\partial u^B_C} \qquad E^{*i}_A = u^i_k v^B_A \frac{\partial}{\partial u^B_k} \tag{VIII.50}
$$

References

[1] Carathéodory, C., "Über die Variationsrechnung bei mehrfachen Integralen," Acta Szeged Sect. Scient. Mathem., **4**, 193-216 (1929).

[2] de Donder, T., "Théorie invariantive du calcul des variations, nouvelle édit.," (Gauthier-Villars, Paris, 1935);

[3] Fulp, R. O., Lawson, J. K. and Norris, L. K., "Geometric prequantization on the spin bundle based on n-symplectic geometry: The Dirac equation", *Int. J. Theor. Phys.* **33**, 1011–1028 (1994); Fulp, R. O., Lawson, J. K. and Norris, L. K., "Generalized symplectic geometry as a covering theory for the Hamiltonian theories of classical particles and fields", *J. Geom. Phys.* **20**, 195-206 (1996).

[4] Kostant, B., In *Lectures in modern analysis III* (ed. C. T. Taam), Lecture notes in mathematics, Vol. 170, (Springer, Berlin, 1970).

[5] Lawson, J. K., "Generalized symplectic geometry for classical fields and spinors", Ph.D. dissertation, Dept. Math., North Carolina State Univ., Raleigh, 1994.

[6] McLean, M. and Norris, L. K., "Covariant Field Theory on Frame Bundles of Fibered Manifolds", J. Math. Phys., **40**, pp. 6808-6823 (2000).

[7] Nijenhuis, A., "Jacobi-type identities for bilinear differential concomitants of certain tensor fields", *Indag. Math.* **17**, 390–403(1955).

[8] Norris, L. K., "Generalized symplectic geometry on the frame bundle of a manifold", Lecture given at the AMS Summer Research Institute on Differential Geometry, 1990, at U.C.L.A; Norris, L. K., "Generalized symplectic geometry on the frame bundle of a manifold", Proc. Symp. Pure Math. **54**, Part 2 (Amer. Math. Soc., Providence RI, 1993) 435–465; Norris, L. K., "Symplectic geometry on T^*M derived from n-symplectic geometry on LM", *J. Geom. Phys.* **13**, 51–78(1994);

[9] Norris, L. K., "Schouten-Nijenhuis Brackets", *J. Math. Phys.* **38**, 2694–2709 (1997) .

[10] Norris, L. K., "n-symplectic Algebra of Observables in Covariant Lagrangian Field Theory", J. Mathematical Physics, **42**, pp. 4827-4845 (2001).

[11] Rund, H., "The Hamilton-Jacobi Theory in the Calculus of Variations", Reprinted Edition, (Robert E. Krieger Publishing Co. Inc., Huntington, New York, 1973).

[12] Saunders, D. J., "The Geometry of Jet Bundles", (Cambridge University Press, Cambridge, 1989).

[13] Souriau, J.-M., "Structure des systèmes dynamiques" (Dunod, Paris, 1970).

[14] Schouten, J. A.,"Über Differentialkomitanten zweier kontravarianter Grossen", *Proc. Kon. Ned. Akad. Wet. Amsterdam* **43**, 449–452 (1940).

[15] Weyl, H., "Geodesic fields in the calculus of variations for multiple integrals," Ann. Math. (2), **36**, 607-629 (1935).

[16] Woodhouse, N. J., "Geometric Quantization", 2nd ed. (Oxford Press, Oxford, 1992).