Schouten-Nijenhuis Brackets†

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Abstract

The Poisson and graded Poisson Schouten-Nijenhuis algebras of symmetric and anti-symmetric contravariant tensor fields, respectively, on an n-dimensional manifold M are shown to be n-symplectic. This is accomplished by showing that both brackets may be defined in a unified way using the n-symplectic structure on the bundle of linear frames LM of M. New results in n-symplectic geometry are presented and then used to give globally defined representations of the Hamiltonian operators defined by the Schouten-Nijenhuis brackets.

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1 Introduction

n-symplectic geometry [3, 4, 12, 15, 16] is the generalized symplectic geometry on the principal bundle of linear frames $LM \to M$ of an n-dimensional manifold M that one obtains by taking the $\mathbb{R}^n$-valued soldering 1-form θ as the generalized symplectic potential. In this paper we describe the explicit relationship of the n-symplectic bracket on LM to the Schouten-Nijenhuis brackets of contravariant tensor fields on M, and use some new results in n-symplectic theory to exhibit globally defined Hamiltonian operators associated with the Schouten-Nijenhuis brackets.

The Schouten-Nijenhuis bracket, first introduced by J. A. Schouten [18] for contravariant tensor fields, was resolved by Nijenhuis [14] into a Lie bracket $(\ ,\ )_{S/N}$ and a graded Lie bracket $[\ ,\ ]_{S/N}$ for the spaces of symmetric $S\mathcal{X}(M) = \oplus_{q=1}^{\infty} S\mathcal{X}^q(M)$ and anti-symmetric $A\mathcal{X}(M) = \oplus_{q=1}^{\infty} A\mathcal{X}^q(M)$ contravariant tensor fields, respectively, on a manifold M. The Schouten-Nijenhuis bracket $(\ ,\ )_{S/N}$ also acts as a derivation on the associative algebra $(S\mathcal{X}(M), \otimes_s)$, thus giving the space $S\mathcal{X}(M)$ the structure of a Poisson algebra [2]. Similarly the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$, which acts as a graded derivation on the associative algebra $(A\mathcal{X}(M), \otimes_a)$, gives the space $A\mathcal{X}(M)$ the structure of a graded Poisson algebra. One is led to ask the question:

> *Are these algebras symplectic? That is to say, are there symplectic structures that one can use to define the Schouten-Nijenhuis brackets?*

This question is geometrical rather than algebraic in spirit. That is to say, we seek to understand the geometrical significance of the Schouten-Nijenhuis brackets for tensor fields on a manifold M, since the algebraic significance of the Schouten-Nijenhuis brackets is now rather well understood. In a manuscript devoted to a study of Poisson structures Bhaskara and Viswanath [2] remark: "*We have observed that this Schouten product is essentially algebraic in nature and that its differential geometric setting is only incidental.*" Their approach was to recast the Schouten-Nijenhuis bracket as a bracket for *multiderivations* of the smooth functions on M. Michor [13] has characterized the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$ as the unique (up to a multiplicative constant) natural concomitant mapping $A\mathcal{X}^q(M)\times A\mathcal{X}^r(M)$ to $A\mathcal{X}^{q+r-1}(M)$. Finally we note that Kosmann-Schwarzbach [9] has described how the Schouten-Nijenhuis brackets are related to the general idea of Loday brackets. Our objective in this paper is to explain the geometrical significance of the Schouten-Nijenhuis brackets in the specific case that they are defined for contravariant tensor fields on a manifold M. The first step in this direction in this paper is to show that the Schouten-Nijenhuis brackets are n-symplectic by showing that both brackets may be defined on a manifold M in a unified way in terms of the n-symplectic bracket on the bundle of linear frames $LM \to M$.

It has long been known, going back to the comment by Nijenhuis [1] in his 1955 paper, that the Schouten-Nijenhuis bracket $(\ ,\ )_{S/N}$ is related to the canonical Poisson bracket of the associated functions on the cotangent bundle T^*M of the manifold. Specifically, suppose that $f \in S\mathcal{X}^q(M)$. Then f defines a real-valued function $\tilde{f}$ on T^*M by the formula

$$\tilde{f}(m,\alpha) = f_m(\underbrace{\alpha,\alpha,\ldots,\alpha}_{q-\text{factors}}) \qquad \forall\ \ (m,\alpha)\in T^*M \tag{1.1}$$

where $m \in M$ and α is a covector in T^*M_m. A function of this type is referred to [21] as a *homogeneous polynomial observable of degree* q on T^*M. Denote by $Poly^q$ the space of homogeneous polynomial observable of degree q on T^*M induced in this way by elements of $S\mathcal{X}^q(M)$, and denote the direct sum of these spaces by $Poly$. Then for $\tilde{f} \in Poly^q$ and $\tilde{g} \in Poly^r$, one may define a bracket operation $(\ ,\ ) : Poly^q \times Poly^r \to Poly^{q+r-1}$ by the formula

$$(\tilde{f},\tilde{g}) = \widetilde{(f,g)}_{S/N} \tag{1.2}$$

where $f \in S\mathcal{X}^q(M)$ and $g \in S\mathcal{X}^r(M)$ are the symmetric contravariant tensor fields

[1] See the **Note added in proof** in [14], p. 397.

on M that define $\tilde{f}$ and $\tilde{g}$, respectively. It is easy to check that this bracket agrees with the restriction of the canonical Poisson bracket on T^*M to $Poly$.

The arguement can also be reversed [21]. One first defines the polynomial observables $Poly$ intrinsically [10] on T^*M relative to the vertical polarization [2]. Then each $f \in Poly^q$ is uniquely related to an element $\sigma(f) \in S\mathcal{X}^q(M)$ on M. One can then define the bracket of $\sigma(f)$ and $\sigma(g)$ to be $\sigma(\{f, g\})$ where $\{\ ,\ \}$ denotes the canonical Poisson bracket on T^*M. One finds that $\sigma(\{f, g\})$ is indeed equal to the Schouten-Nijenhuis bracket $(\sigma(f), \sigma(g))_{S/N}$. For these reasons we may say that the Schouten-Nijenhuis bracket $(\ ,\ )_{S/N}$ is symplectic. Note, however, that there is no possibility of lifting the Schouten-Nijenhuis bracket for anti-symmetric tensor fields to define a bracket on T^*M since the right hand side of (1.1) vanishes identically if the tensor f is anti-symmetric. In this paper we will show that both brackets may be considered as **n-symplectic** brackets in that both of the Schouten-Nijenhuis brackets on M can be defined in a unified way using the n-symplectic bracket on LM. The conclusion to be drawn is that the Schouten-Nijenhuis brackets for contravariant tensor fields on a manifold M may be thought of as remnants of the n-symplectic structure on LM, and the geometrical significance of the Schouten-Nijenhuis brackets is that they reflect the two independent degrees of freedom in specifying, for example, a rank p contravariant tensor field, namely the freedom to choose an arbitrary frame (the LM degree of freedom) and the freedom to choose components (the $T^p\mathbb{R}^n$ degree of freedom). These two independent degrees of freedom are tied together by the $GL(n, \mathbb{R})$ tensorial action, and this all shows up explicitly when the rank p contravariant tensor bundle is thought of as the vector bundle

$$LM \times_{GL(n,\mathbb{R})} T^p\mathbb{R}^n$$

[2]The intrinsic definition yields the non-homogeneous polynomials. One can then define the subspace of homogeneous polynomials of degree q to be composed of the degree q polynomials that satisfy the equation $\mathcal{E}(f) = -qf$, where $\mathcal{E}$ is the Euler vector field on T^*M defined by $\vartheta = -\mathcal{E} \rfloor d\vartheta$, and ϑ is the $\mathbb{R}$-valued canonical 1-form on T^*M.

associated to LM.

The second goal of the paper is to develop explicit globally defined formulas for the Hamiltonian operators defined by the Schouten-Nijenhuis brackets in terms of geometrical quantities that can be integrated to yield integral curves and their generalizations. Following the ideas in [2] we define a Hamiltonian operator X_f for $f \in S\mathcal{X}^q(M)$ by the formula

$$X_f(g) = (f, g)_{S/N} \qquad\qquad \forall\, g \in S\mathcal{X}(M) \tag{1.3}$$

These Hamiltonian operators, which generalize the Hamiltonian vector fields on a symplectic manifold, form an infinite dimensional vector space and clearly should have an algebraic structure related to the Poisson algebra $(S\mathcal{X}(M), (\ ,\ )_{S/N})$. The question arises as to whether one can find an explicit formula for X_f in terms of tensors and other geometric objects and operations. There is a parallel question for the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$. Michor [13] obtains invariant formulas for $[\ ,\ ]_{S/N}$ (see also Tulczyjew [20]) and hence implicitly, invariant formulas for the associated Hamiltonian operators. Our goal is to develop this further by obtaining explicit representations of the Hamiltonian operators in terms of familar geometric quantities. The basic idea can be described as follows.

We recall [14] that the original definition of the Schouten-Nijenhuis bracket was given in terms of local coordinates on M as follows. Consider, for example, the symmetric case. Suppose A and B are symmetric rank p and q contravariant tensor fields, respectively, on an n-dimensional manifold M. Let their components with respect to some chart (x^i) be denoted by $A^{i_1 i_2 \ldots i_p}$ and $B^{k_1 k_2 \ldots k_q}$. Then the Schouten bracket of A and B is a rank $p+q-1$ symmetric contravariant tensor field $(A, B)_{S/N}$ with components

$$(A, B)_{S/N}^{i_1 \ldots i_{p-1} j k_1 \ldots k_{q-1}} = p A^{l(i_1 i_2 \ldots i_{p-1}} \frac{\partial}{\partial x^l} B^{j k_1 \ldots k_{q-1})} - q B^{l(k_1 \ldots k_{q-1}} \frac{\partial}{\partial x^l} A^{j i_1 \ldots i_{p-1})} \tag{1.4}$$

where the round brackets ( ) around indices denotes symmetrization over those indices. This formula has the remarkable property that the right-hand-side in invariant under the substitution $\frac{\partial}{\partial x^i} \longrightarrow \nabla_i$ where ∇ denotes the operation of covariant differentiation with respect to any torsion-free linear connection. It is in fact this property that guarantees that the right-hand-side of this formula indeed gives the components of a rank $p+q-1$ tensor field. Thus for each choice of a symmetric linear connection, one can write down the above formula in a globally defined, invariant way without reference to any coordinate system. One should then also be able to extract an invariant definition for X_f on M, which would certainly involve the algebraic concepts of derivations [13] and multiderivations [2]. We prefer instead to seek representations in terms of geometrical quantities, like vector fields, that can be integrated to yield geometrical information. Thus, rather than pursuing this directly, we will instead construct the n-symplectic Hamiltonian operators on LM, once we have established the fact that the Schouten-Nijenhuis brackets are n-symplectic. These Hamiltonian operators on LM turn out to be equivalence classes of vector-valued vector fields, where the equivalence classes reflect a certain n-symplectic gauge freedom. We will show that each choice of a torsion-free linear connection on LM breaks the gauge symmetry and selects unique representatives from the equivalence classes. This symmetry breaking by torsion-free connections is the n-symplectic characterization of the substitution freedom $\frac{\partial}{\partial x^i} \longrightarrow \nabla_i$ discussed above. In order to illustrate the significance of this symmetry breaking we apply the theory to the "free observer system" in a fixed curved spacetime. We show that if one selects the Levi-Civita connection defined by the spacetime metric itself, then integration of the time-like Hamiltonian vector field of the "free observer Hamiltonian" on LM yields freely falling observers in the spacetime, i.e. non-rotating observers moving along geodesics of the Levi-Civita connection. If instead one fixes the n-symplectic gauge by selecting an arbitrary torsion-free connection, then one still finds observers moving along the spacetime geodesics of the metric connection, but these observers experience "generalized rotational forces" that are generated by the non-metricity of the chosen connection.

The structure of the paper is as follows. In Section 2 we show that both Schouten-

Nijenhuis brackets are n-symplectic. In Section 3 we first present a brief review of the n-symplectic Hamiltonian operators associated with the vector-valued tensorial functions on LM. These n-symplectic Hamiltonian operators are equivalence classes of sets of vector fields. We then present new results that show how to select representatives of the n-symplectic Hamiltonian operators for each choice of a torsion-free linear connection on LM. In Section 4 we apply the theory to the "free observer system" discussed above. Finally in Section 5 we present conclusions about the results of this paper and the relationship of n-symplectic geometry, and hence Schouten-Nijenhuis brackets, to other more standard symplectic type theories. There is also an appendix which contains some facts about n-symplectic geomtry that are needed earlier in the paper.

For later reference we collect together here much of the notation that will be used throughtout the paper.

NOTATION

1. $\mathcal{X}^p(M)$ dentoes the space of smooth rank p contravariant tensor fields on M, with $p \geq 1$.

2. $T^p(LM)$ denotes the space of smooth $GL(n, \mathbf{R})$-tensorial functions on LM, with ranges $(\otimes)^p\mathbf{R}^n$.

3. $S\mathcal{X}^p(M)$ and $A\mathcal{X}^p(M)$ denote the spaces of smooth rank p symmetric and anti-symmetric contravariant tensor fields, respectively, on M, with $p \geq 1$.

4. $S\mathcal{X}(M) = \oplus_{p=1}^{\infty} S\mathcal{X}^p(M)$ and $A\mathcal{X}(M) = \oplus_{p=1}^{\infty} A\mathcal{X}^p(M)$.

5. $\otimes_s$ denotes the symmetric and $\otimes_a$ the anti-symmetric tensor product.

6. $ST^p(LM)$ and $AT^p(LM)$ denote the spaces of smooth $GL(n, \mathbf{R})$-tensorial functions on LM, with ranges $(\otimes_s)^p\mathbf{R}^n$ and $(\otimes_a)^p\mathbf{R}^n$, respectively.

7. $ST(LM) = \oplus_{p=1}^{\infty} ST^p(LM)$ and $AT(LM) = \oplus_{p=1}^{\infty} AT^p(LM)$.

2 The n-symplectic structure on LM

The $GL(n,\mathbb{R})$- principal fiber bundle of linear frames LM of an n-dimensional manifold M supports a canonically defined $\mathbb{R}^n$-valued 1-form, the so-called soldering 1-form θ (see the appendix). As defined in [15], n-symplectic geometry is the generalized symplectic geometry on LM that one obtains by taking $d\theta$ as the generalized symplectic form. Some facts about n-symplectic geometry are listed in the appendix, and the interested reader may find more details in [15, 16].

We recall [8] that a rank p contravariant tensor field f on an n-dimensional manifold M is uniquely related to a $\otimes^p(\mathbb{R}^n)$-valued tensorial function $\hat{f}$ on LM as follows. Represent a point $u \in LM$ by the pair (m, e_i) where (e_i) , $i = 1, 2, \ldots, n$ denotes a linear frame at $m \in M$. Then for each $p \geq 1$, one may consider u as the linear map

$$u : \otimes^p\mathbb{R}^n \to T^pM_m \quad , \quad u(\xi^{i_i i_2 \ldots i_p}) = \xi^{i_i i_2 \ldots i_p} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p} \tag{2.5}$$

with inverse mapping

$$u^{-1} : T^pM_m \to \otimes^p\mathbb{R}^n \quad , \quad u^{-1}(\xi) = (\xi(e^{i_1}, e^{i_2}, \ldots, e^{i_p})) \tag{2.6}$$

The domain of u and the range of u^{-1} specialize to $(\otimes_s)^p\mathbb{R}^n$ if f is symmetric, and to $(\otimes_a)^p\mathbb{R}^n$ if f is anti-symmetric.

Let $\pi : LM \to M$ be the canonical projection $\pi(m, e_i) = m$. Given $f \in \mathcal{X}^p(M)$ one defines $\hat{f} \in T^p(LM)$ on LM by the formula

$$\hat{f}(u) = u^{-1}(f(\pi(u))) \tag{2.7}$$

One shows [8] that such a function $\hat{f}$ satisfies the tensorial transformation law

$$R_g^*(\hat{f}) = g^{-1} \cdot \hat{f} \ , \qquad \forall g \in GL(n,\mathbb{R}) \tag{2.8}$$

where R_g denotes right translation on LM by $g \in GL(n,\mathbb{R})$, and the dot on the right hand side denotes the standard action of $GL(n,\mathbb{R})$ on $(\otimes_s)^p\mathbb{R}^n$. Conversely, given $\hat{f} \in T^p(LM)$ on LM, one defines $f \in \mathcal{X}^p(M)$ on M by the formula

$$f(m) = u(\hat{f}(u)) \tag{2.9}$$

where u is any point in LM such that $\pi(u) = m$. The tensorial character of $\hat{f}$ guarantees that the right-hand-side of this last equation is independent of which $u \in \pi^{-1}(m)$ one uses, so that f is well-defined.

Now let $\hat{f} = (\hat{f}^{i_1 \cdots i_q}) \in ST^q(LM)$. Then the associated n-symplectic Hamiltonian operator $[\![X_{\hat{f}}]\!]$ is determined by the n-symplectic structure equation [3]

$$d\hat{f}^{i_1 \ldots i_q} = -q! X_{\hat{f}}{}^{(i_1 \ldots i_{q-1}} \lrcorner\, d\theta^{i_q)} \quad . \tag{2.10}$$

where the round brackets around superscripts denote symmetrization over the inclosed indices. This equations determines an equivalence class $[\![X_{\hat{f}}]\!]$ of $\binom{n+q-2}{q-1}$ vector fields $(X_{\hat{f}}{}^{i_1 \ldots i_{q-1}})$, and the explicit local coordinate form of a representative of an equivalence class of vector fields is given in (A.9) below. The n-symplectic Hamiltonian operators turn out to be equivalence classes of vector fields because the symmetrization of the indices in (2.10) introduces a certain degeneracy (cf. appendix equations (A.7) and (A.10)). Nontheless, one can define a bracket operation $\{\ ,\ \} : ST^q(LM) \times ST^r(LM) \rightarrow ST^{q+r-1}(LM)$ by the formula

$$\{\hat{f}, \hat{g}\}^{i_1 i_2 \ldots i_{q+r-1}} = q! X_{\hat{f}}{}^{(i_1 i_2 \ldots i_{q-1}} \left(\hat{g}^{i_q i_{q+1} \ldots i_{q+r-1})}\right) \tag{2.11}$$

for $\hat{f} \in ST^q(LM)$ and $\hat{g} \in ST^r(LM)$. In this formula $(X_{\hat{f}}{}^{i_1 i_2 \ldots i_{q-1}})$ is any representative of the equivalence class $[\![X_{\hat{f}}]\!]$. The bracket so defined is easily shown to be independent of the choice of representatives and has all the properties of a Poisson bracket. In particular the bracket acts as a derivation on the associative algebra $(ST, \otimes_s)$.

Theorem 2.1 *The space ST of symmetric tensorial functions on LM is a Poisson algebra with respect to the n-symplectic bracket $\{\ ,\ \}$ defined in (2.11).*

One may now define a bracket on elements of $S\mathcal{X}(M)$ on M as follows. For $f \in S\mathcal{X}^q(M)$, $g \in S\mathcal{X}^r(M)$ define (f, g) to be the unique element of $S\mathcal{X}^{q+r-1}(M)$

[3]This equation is the n-symplectic generalization of the structure equation $df = -X_f \lrcorner\, \omega$ on a symplectic manifold (M, ω) for $f \in C^\infty(M)$.

determined by

$$\widehat{(f,g)} = \{\hat{f}, \hat{g}\} \tag{2.12}$$

It is known [15] that **this bracket is the Schouten-Nijenhuis bracket of f and g.** Moreover, in [15] it is shown that $(ST, \{\ ,\ \})$ is a proper sub-algebra of the full n-symplectic Poisson algebra of symmetric tensor-valued functions on LM.

We have the result that the Schouten-Nijenhuis bracket for symmetric contravariant tensor fields on a manifold is both symplectic and n-symplectic, as it can be defined by both structures. However, as mentioned above, it does not seem possible to define the Schouten-Nijenhuis bracket for anti-symmetric contravariant tensor fields in terms of the symplectic structure on T^*M. On the otherhand, we now show that the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$ is also n-symplectic.

Let $\hat{f} = (\hat{f}^{i_1\ldots i_q}) \in AT^q(LM)$. Then the associated n-symplectic Hamiltonian operator $[\![X_{\hat{f}}]\!]$ is determined by the n-symplectic structure equation

$$d\hat{f}^{i_1\ldots i_q} = -q! X_{\hat{f}}{}^{[i_1\ldots i_{q-1}} \lrcorner\ d\theta^{i_q]} \quad . \tag{2.13}$$

where the square brackets around indices denote anti-symmetrization over the encloded indices. This equation determines an equivalence class $[\![X_{\hat{f}}]\!]$ of $\binom{n}{q-1}$ vector fields $(X_{\hat{f}}{}^{i_1\ldots i_{q-1}})$, and the explicit local coordinate form of an equivalence class of vector fields is given in (A.14) below. These Hamiltonian operators again turn out to be equivalence classes of vector fields because the anti-symmetrization of the indices in (2.13) also introduces a degeneracy (cf. appendix equations (A.12) and (A.15)). As in the symmetric case one can define a bracket operation $\{\ ,\ \} : AT^q(LM) \times AT^r(LM) \to AT^{q+r-1}(LM)$ by the formula

$$\{\hat{f}, \hat{g}\}^{i_1 i_2 \ldots i_{q+r-1}} = q! X_{\hat{f}}{}^{[i_1 i_2 \ldots i_{q-1}} \left(\hat{g}^{i_q i_{q+1}\ldots i_{q+r-1}]}\right) \tag{2.14}$$

for $\hat{f} \in AT^q(LM)$ and $\hat{g} \in AT^r(LM)$. In this formula $(X_{\hat{f}}{}^{i_1 i_2 \ldots i_{q-1}})$ is any representative of the equivalence class $[\![X_{\hat{f}}]\!]$. The bracket so defined is easily shown [15] to be independent of the choice of representatives.

Theorem 2.2 *Let $\hat{f} \in AT^p(LM)$, $\hat{g} \in AT^q(LM)$, and $\hat{h} \in AT^r(LM)$. Then the bracket operation defined in (2.14) has the following properties:*

$$\begin{array}{llll}
\textbf{(a)} & \{\hat{f},\hat{g}\} & = -(-1)^{(p-1)(q-1)}\{\hat{g},\hat{f}\} & \\
\textbf{(b)} & 0 & = (-1)^{(p-1)(r-1)}\{\ \hat{f},\{\hat{g},\hat{h}\}\} + (-1)^{(p-1)(q-1)}\{\ \hat{g},\{\hat{h},\hat{f}\}\} & \\
 & & \quad +(-1)^{(q-1)(r-1)}\{\hat{h},\{\hat{f},\hat{g}\}\} & \\
\textbf{(c)} & \{\hat{f},\hat{g}\otimes_a\hat{h}\} & = \{\hat{f},\hat{g}\}\otimes_a\hat{h} + (-1)^{(p-1)(q)}\hat{g}\otimes_a\{\hat{f},\hat{h}\} & (2.15)
\end{array}$$

Remark The space AT is the direct sum $AT = \oplus_{p=1}^{\infty} AT^p(LM)$ of the rank p anti-symmetric tensor-valued tensorial functions on LM. We assign the degree $|\hat{f}|$ of an element $\hat{f} \in AT^p(LM)$ as follows:

$$|\hat{f}| = \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{if } p \text{ is even} \end{cases} \tag{2.16}$$

With this grading Theorem 2.2 shows that $(AT, \{\ ,\ \})$ is a Z_2 graded Lie algebra with the bracket acting as a graded derivation on $(AT, \otimes_a)$. Hence we have the following theorem.

Theorem 2.3 *The space AT of anti-symmetric tensorial functions on LM is a graded Poisson algebra with respect to the n-symplectic bracket $\{\ ,\ \}$ defined in (2.14).*

One may now define a bracket on elements of $A\mathcal{X}(M)$ on M as follows. For $f \in A\mathcal{X}^q(M)$, $g \in A\mathcal{X}^r(M)$ define $[f, g]$ to be the unique element of $A\mathcal{X}^{q+r-1}(M)$ determined by

$$\widehat{[f, g]} = \{\hat{f}, \hat{g}\} \tag{2.17}$$

It is known [15] that **this bracket is, up to a sign, the Schouten-Nijenhuis bracket $[f, g]_{S/N}$ of f and g.** Specifically, we have the following theorem.

Theorem 2.4 *Let $f \in A\mathcal{X}^q(M)$ and $g \in A\mathcal{X}^r(M)$. Then the Schouten-Nijenhuis bracket $[f,g]_{S/N}$ and the bracket $[f,g]$ defined above in (2.17) by the n-symplectic structure on LM are related by*

$$[f,g] = (-)^{|f|}[f,g]_{S/N} \tag{2.18}$$

Remark This sign difference is simply a consequence of the different orderings of the anti-symmetric indices in the original definition given by Nijenhuis [14] and the definition (2.14), and either bracket can be redefined to absorb this factor. For example, if $\{\ ,\ \}$ in (2.14) is replaced with a new bracket $\{\ ,\ \}_0$ defined by

$$\{\hat{f},\hat{g}\}_0^{i_1 i_2 \ldots i_{q+r-1}} = (-1)^{q-1} q! X_{\hat{f}}{}^{[i_1 i_2 \ldots i_{q-1}} \left(\hat{g}^{i_q i_{q+1} \ldots i_{q+r-1}]}\right) \tag{2.19}$$

then the bracket induced on M by using $\{\ ,\ \}_0$ on the right-hand-side in (2.17), and the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$ coincide. We note that Michor [13] also found it more natural to define the Schouten-Nijenhuis bracket for anti-symmetric fields with a sign that is opposite from the sign in the original definition [14].

3 Hamiltonian Operators

In order to discuss the n-symplectic Hamiltonian operators efficiently, it is convenient to use a multi-index notation. In the following we let $I_q = i_1 \ldots i_{q-1}$ and $J_r = j_1 \ldots j_{r-1}$ for $q, r \geq 1$. Then we can use the notation $X^{I_q} = X^{i_1 \cdots i_{q-1}}$ to denote the individual vector fields that are contained in a set of vector fields $(X^{i_1 \cdots i_{q-1}})$. In particular, we denote the equivalence classes of vector fields determined by $\hat{f} \in ST^q$ and $\hat{g} \in ST^r$ by $[\![X_{\hat{f}}]\!] \equiv [\![X_{\hat{f}}^{I_q}]\!]$ and $[\![X_{\hat{g}}]\!] \equiv [\![X_{\hat{g}}^{J_r}]\!]$, respectively. In addition we denote by $HO(ST^q)$ the vector space of Hamiltonian operators $[\![X_{\hat{f}}]\!]$ for $\hat{f} \in ST^q$, and denote the direct sum of the vector spaces $HO(ST^q)$ by $HO(ST)$. Similarly, $HO(AT^q)$ will denote the space of n-symplectic Hamiltonian operators $[\![X_{\hat{f}}]\!]$ for $\hat{f} \in AT^q$, and $HO(AT)$ will denote the direct sum of the vector spaces $HO(AT^q)$.

Define a bracket operation $[\ ,\ ]: HO(ST^q) \times HO(ST^r) \to HO(ST^{q+r-1})$ by

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = \left([X_{\hat{f}}{}^{(I_q}, X_{\hat{g}}{}^{J_r)}]\right) \tag{3.20}$$

where the bracket on the right-hand-side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives, and the indices have been symmetrized. One shows that for any set of representatives $(X_{\hat{f}}{}^{I_q})$ of $[\![X_{\hat{f}}]\!]$,

$$\frac{q!\,r!}{(q+r-1)!}\left([X_{\hat{f}}{}^{(I_q}\ ,\ X_{\hat{g}}{}^{J_r)}]\right) \in [\![X_{\{\hat{f},\hat{g}\}}]\!] \tag{3.21}$$

Thus the bracket $[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]$ is well-defined, and we write

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = \frac{(q+r-1)!}{q!\,r!}[\![\hat{X}_{\{\hat{f},\hat{g}\}}]\!] \quad . \tag{3.22}$$

Moreover, it is known [15] that the bracket defined in (3.20) is anti-symmetric and satisfies the Jacobi identity.

Theorem 3.1 *The vector space $HO(ST)$ of n-symplectic Hamiltonian operators (equivalence classes of Hamiltonian vector fields) on LM is a Lie algebra with respect to the bracket defined in (3.20).*

There is a parallel development for the Hamiltonian operators defined by the anti-symmetric tensor-valued functions on LM. Define a bracket operation $[\ ,\ ]$: $HO(AT^q) \times HO(AT^r) \to HO(AT^{q+r-1})$ by

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = \left([X_{\hat{f}}{}^{[I_q}, X_{\hat{g}}{}^{J_r]}]\right) \tag{3.23}$$

where the bracket on the right-hand-side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives, and the indices have been anti-symmetrized. As in the symmetric case one shows that for any set of representatives $(X_{\hat{f}}{}^{I_q})$ of $[\![X_{\hat{f}}]\!]$,

$$\frac{q!\,r!}{(q+r-1)!}\left([X_{\hat{f}}{}^{(I_q}\ ,\ X_{\hat{g}}{}^{J_r)}]\right) \in [\![X_{\{\hat{f},\hat{g}\}}]\!] \tag{3.24}$$

Thus the bracket $[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]$ is also well-defined, and as in (3.22)we write

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = \frac{(q+r-1)!}{q!\, r!} [\![\hat{X}_{\{\hat{f},\hat{g}\}}]\!] \quad . \tag{3.25}$$

As one would expect the algebraic properties of this bracket mirror the properties listed in Theorem 2.2 above for the n-symplectic bracket of anti-symmetric tensor-valued functions.

Theorem 3.2 *Let $\hat{f} \in AT^p(LM)$, $\hat{g} \in AT^q(LM)$, and $\hat{h} \in AT^r(LM)$, and let $[\![\hat{X}_{\hat{f}}]\!]$, $[\![\hat{X}_{\hat{g}}]\!]$ and $[\![\hat{X}_{\hat{h}}]\!]$ denote the corresponding n-symplectic Hamiltonian operators. Then the bracket operation defined in (3.23) has the following properties:*

$$\begin{aligned}
&(\mathbf{a}) \quad [[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] &&= -(-1)^{(p-1)(q-1)}[[\![\hat{X}_{\hat{g}}]\!], [\![\hat{X}_{\hat{f}}]\!]] \\
&(\mathbf{b}) \quad 0 &&= (-1)^{(p-1)(r-1)}[[\![\hat{X}_{\hat{f}}]\!], [[\![\hat{X}_{\hat{g}}]\!], [\![\hat{X}_{\hat{h}}]\!]]] + (-1)^{(p-1)(q-1)}[[\![\hat{X}_{\hat{g}}]\!], [[\![\hat{X}_{\hat{h}}]\!], [\![\hat{X}_{\hat{f}}]\!]]] \\
& && \quad +(-1)^{(q-1)(r-1)}[[\![\hat{X}_{\hat{h}}]\!], [\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]]
\end{aligned} \tag{3.26}$$

These properties of the n-symplectic bracket for the Hamiltonian operators corresponding to elements of AT yield the following theorem.

Theorem 3.3 *The vector space $HO(AT)$ of n-symplectic Hamiltonian operators (equivalence classes of Hamiltonian vector fields) on LM is a Z_2 graded Lie algebra with respect to the bracket defined in (3.23).*

The point to be emphasized here is that the Hamiltonian operators for the Poisson algebra of symmetric and anti-symmetric tensorial functions on LM are equivalence classes of sets of vector fields, rather than sets of vector fields. This is a consequence of the symmetrization and anti-symmetrizations of the indices in the structure equations (2.10) and (2.13). This fact is also related to the observation [14] that the local coordinate formulas for the Schouten-Nijenhuis brackets are invariant under the substitution $\partial_i \longrightarrow \nabla_i$ mentioned earlier. This relationship follows from the following theorem.

Theorem 3.4 *Let B_i, $i = 1, 2, \ldots, n$ be the standard horizontal vector fields of a symmetric linear connection 1-form ω on the bundle of linear frames LM of an n-dimensional manifold M. Let E^{i*}_j, $i, j = 1, 2, \ldots, n$ denote the fundamental vertical vector fields on LM defined by the standard basis (E^i_j) of $gl(n, \mathbb{R})$. For $\hat{f} = (\hat{f}^{iI_q}) \equiv (\hat{f}^{i_1 \ldots i_q}) \in ST^q(LM)$ define a set of vector fields*

$$\widetilde{X_{\hat{f}}}^{I_q} = \frac{1}{(q-1)!}\left(\hat{f}^{jI_q}\right) B_j + \frac{1}{q!}\left(D_k \hat{f}^{I_q j}\right) E^{k*}_j \tag{3.27}$$

where $D_k = B_k \rfloor D^\omega$ and D^ω denotes exterior covariant differentiation with respect to ω. Then

$$\left(\widetilde{X_{\hat{f}}}^{I_q}\right) \in [\![X_{\hat{f}}]\!] \tag{3.28}$$

Proof According to appendix equation (A.9) the local coordinate form of a representative of $[\![X_{\hat{f}}]\!]$ can be given by specifying functions $T^{I_q b}_a$ that satisfy (A.10). Let ω be a torsion-free connection 1-form on LM. Then in the local canonical coordinates (x^i, π^j_k) defined in appendix equation (A.2) the associated standard horizontal vector fields defined by ω take the form

$$B_j = (\pi^{-1})^k_j \left(\frac{\partial}{\partial x^k} + \Gamma^l_{ka} \pi^i_l \frac{\partial}{\partial \pi^i_a}\right) \tag{3.29}$$

where the (Γ^l_{ka}) are the local coordinate components of the connection ω. In addition the local coordinate formulas for the fundamental vertical vector fields on LM are

$$E^{i*}_j = -\pi^i_k \frac{\partial}{\partial \pi^j_k} \tag{3.30}$$

Substituting these expressions into (3.27) one shows that (3.27) reduces to the form given in (A.9) where the functions $T^{I_q b}_a$ satisfy (A.10). ∎

We have the result that each choice of a torsion-free linear connection ω on LM yields a globally defined Hamiltonian operator for each $\hat{f} \in ST$, and hence selects a subspace $HO^\omega(ST) \subset HO(ST)$. The question remains whether or not $HO^\omega(ST)$ is a sub-algebra under the bracket defined above in (3.23). While (3.24) clearly is still satisfied, a direct calculation shows that

$$[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}] \not\propto \widetilde{X}_{\{\hat{f},\hat{g}\}} \tag{3.31}$$

The reason is that while the vertical component of $\widetilde{X}_{\{\hat{f},\hat{g}\}}$ is clearly symmetric (see (3.27)), the vertical component of $[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]$ is not symmetric. Fortunately, it is possible to modify the definition (3.20) so that the bracket closes on the set $HO^{\omega}(ST)$ by enforcing symmetry on the vertical components. This can be done globally on LM since the n components θ^i of the soldering 1-form and the n^2 components ω^i_j of a connection 1-form ω together define a global basis of 1-forms on LM.

Definition 3.5 *For $\hat{f} \in ST^q(LM)$ and $\hat{g} \in ST^r(LM)$, let $\widetilde{X}_{\hat{f}} = (\widetilde{X}^{I_q}_{\hat{f}})$ and $\widetilde{X}_{\hat{g}} = (\widetilde{X}^{J_r}_{\hat{g}})$ be as in (3.27) above for some torsion-free connection 1-form $\omega = (\omega^i_j)$. Define a bracket operation $[\ ,\ ]_\star : HO^{\omega}(ST^q) \times HO^{\omega}(ST^r) \to HO^{\omega}(ST^{q+r-1})$ by $[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]_\star = (\Big([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_qJ_r}_\star\Big)$, where*

$$[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_qJ_r}_\star = \Big([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_qJ_r} \lrcorner\, \theta^i\Big)\, B_i + \Big([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{(I_qJ_r} \lrcorner\, \omega^{i)}_j\Big)\, E^{j*}_i \tag{3.32}$$

and where the bracket on the right-hand side is the bracket defined in (3.20).

Remark The symmetrization of the indices in the vertical component is the only difference between this new bracket and the bracket defined in (3.20). The tedious but straigtforward proof of the following theorem is omitted.

Theorem 3.6 *The bracket $[\ ,\ ]_\star$ defined above in (3.32) satisfies*

$$[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]_\star = \frac{(q+r-1)!}{q!r!}\widetilde{X}_{\{\hat{f},\hat{g}\}} \tag{3.33}$$

Corollary 3.7 *The space $(HO^{\omega}(ST), [\ ,\ ]_\star)$ is a Lie algebra.*

For completeness, we quote without proof the analogous results for the Hamiltonian operators for anti-symmetric tensorial functions on LM.

Theorem 3.8 *Let B_i, $i = 1, 2, \ldots, n$ be the standard horizontal vector fields of a symmetric linear connection 1-form ω on the bundle of linear frames LM of an n-dimensional manifold M. Let E^{i*}_j, $i, j = 1, 2, \ldots, n$ denote the fundamental vertical vector fields on LM defined by the standard basis (E^i_j) of $gl(n, \mathbb{R})$. For $\hat{f} = (\hat{f}^{iI_q}) \equiv (\hat{f}^{i_1 \ldots i_q}) \in AT^q(LM)$ define a set of vector fields*

$$\widetilde{X_{\hat{f}}}^{I_q} = \frac{1}{(q-1)!} \left(\hat{f}^{jI_q}\right) B_j + \frac{1}{q!} \left(D_k \hat{f}^{I_q j}\right) E^{k*}_j \tag{3.34}$$

where $D_k = B_k \lrcorner\, D^\omega$ and D^ω denotes exterior covariant differentiation with respect to ω. Then

$$\left(\widetilde{X_{\hat{f}}}^{I_q}\right) \in [\![X_{\hat{f}}]\!] \tag{3.35}$$

Definition 3.9 *For $\hat{f} \in AT^q(LM)$ and $\hat{g} \in AT^r(LM)$, let $\widetilde{X}_{\hat{f}} = (\widetilde{X}^{I_q}_{\hat{f}})$ and $\widetilde{X}_{\hat{g}} = (\widetilde{X}^{J_r}_{\hat{g}})$ be as in (3.34) above for some torsion-free connection 1-form $\omega = (\omega^i_j)$. Define a bracket operation $[\ ,\ ]_\star : HO^\omega(AT^q) \times HO^\omega(AT^r) \to HO^\omega(AT^{q+r-1})$ by $[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]_\star = (\left([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_q J_r}_\star\right)$, where*

$$[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_q J_r}_\star = \left([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{I_q J_r} \lrcorner\, \theta^i\right) B_i + \left([\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]^{[I_q J_r} \lrcorner\, \omega^{i]}_j\right) E^{j*}_i \tag{3.36}$$

and where the bracket on the right-hand side is the bracket defined in (3.23).

Remark The anti-symmetrization of the indices in the vertical component is the only difference between this new bracket and the bracket defined in (3.23).

Theorem 3.10 *The bracket $[\ ,\ ]_\star$ defined above (3.36) satisfies*

$$[\widetilde{X}_{\hat{f}}, \widetilde{X}_{\hat{g}}]_\star = \frac{(q+r-1)!}{q!r!} \widetilde{X}_{\{\hat{f}, \hat{g}\}} \tag{3.37}$$

Corollary 3.11 *The space $(HO^\omega(AT), [\ ,\ ]_\star)$ is a graded Lie algebra.*

4 Applications: The free observer system in spacetime.

We have just seen that for each choice of a torsion-free linear connection one obtains unique representatives of the equivalence classes of Hamiltonian operators for both symmetric and anti-symmetric tensorial functions on LM. We will refer to this freedom to choose a symmetric connection as an "n-symplectic gauge freedom", and will refer to a choice of symmetric connection as a "choice of n-symplectic gauge". In order to gain geometrical insight into the meaning of the n-symplectic gauge we apply the theory to the simpliest of all possible systems, namely the *free observer system* in a 4-dimensional spacetime manifold M with metric tensor field $\vec{g}$.

We first recall that the contravariant form of the metric tensor defines on T^*M the *free particle Hamiltonian* $\mathcal{H} = \frac{1}{2m}\tilde{g}$, where m is the mass of the particle and $\tilde{g}$ is the function on T^*M defined by $\vec{g}$ as in (1.1). The integral curves of the Hamiltonian vector field defined by $\mathcal{H}$ project to the geodesics of the Levi-Civita connection defined by the metric tensor. The *free observer system* alluded to above is the observable defined on LM by $\vec{g}$, and is the analogue of the *free particle Hamiltonian.*

The generalized Hamiltonian of the *free observer system* is thus the symmetric observable $\hat{H} = \hat{g}$ where $\hat{g} = (\hat{g}^{ij}) \in ST^2(LM)$ is the tensorial function defined on LM as in (2.7). (For simplicity we have dropped the multiplicative constant $\frac{1}{2m}$.) The n-symplectic structure equation (2.10) determines the Hamiltonian operator $[\![X_{\hat{g}}]\!]$, each member of which is an $\mathbf{R}^n$-valued vector field on LM. In order to find integrals of this Hamiltonian operator we select an n-symplectic gauge, namely an arbitrary torsion-free linear connection ω on LM. Then by Theorem (3.4) we obtain the following unique Hamiltonian operator for $\hat{g}$:

$$\widetilde{X_{\hat{g}}}^{i} = \hat{g}^{ij}B_j + \frac{1}{2}\left(D_k\hat{g}^{ij}\right)E_j^{k*}\ , \quad i = 0,1,2,3 \tag{4.38}$$

In this specific case we use standard notation to rewrite this equation as

$$\widetilde{X_{\hat{g}}}^{i} = \hat{g}^{ij} B_j + \frac{1}{2} \left(\hat{Q}_k{}^{ij} \right) E_j^{k*} \ , \quad i = 0, 1, 2, 3 \tag{4.39}$$

where $\hat{Q}_k{}^{ij} = D_k \hat{g}^{ij}$ is the *non-metricity* [19] of the n-symplectic gauge ω. Let us now find the integral curve of $\widetilde{X_{\hat{g}}}^{0}$ that starts at the initial frame $u_0 \in LM$, where u_0 is such that $\widetilde{X_{\hat{g}}}^{0}(u_0)$ projects to a time-like vector at $\pi(u_0)$. Substituting (3.29) and (3.30) into (4.39) we obtain the following system of equations for the integral curves of $\widetilde{X_{\hat{g}}}^{0}$ in the local canonical coordinates defined in the appendix:

$$\begin{aligned} \dot{x}^b &= g^{ab} \pi_a^0 \\ \dot{\pi}_a^j &= -\tfrac{1}{2} \left(Q_a{}^{bl} - 2\Gamma^l_{ak} g^{bk} \right) \pi_b^0 \pi_l^j \end{aligned} \tag{4.40}$$

From the definition of the generalized momentum coordinates π_j^i given in appendix equation (A.2) we see that the second of the above equations is a transport equation for a coframe rather than frame.

This system of equations splits into the following two sets of equations:

$$\begin{aligned} \dot{x}^b &= g^{ab} \pi_a^0 \\ \dot{\pi}_a^0 &= -\tfrac{1}{2} \left(Q_a{}^{bl} - 2\Gamma^l_{ak} g^{bk} \right) \pi_b^0 \pi_l^0 \end{aligned} \tag{4.41}$$

and, for $\alpha = 1, 2, 3$,

$$\dot{\pi}_a^\alpha = -\frac{1}{2} \left(Q_a{}^{bl} - 2\Gamma^l_{ak} g^{bk} \right) \pi_b^0 \pi_l^\alpha \tag{4.42}$$

Consider first the coupled system (4.41). Using the relationship [19] between the connection coefficients Γ^l_{ak}, the Christoffel symbols $\{^{\,i}_{jk}\}$, and the non-metricity $Q_a{}^{bl}$, one finds that the non-metricity and the n-symplectic gauge Γ^l_{ak} cancel out, and that (4.41) reduces to the equation for the geodesic of the Levi-Civita connection, namely

$$\frac{d^2 x^i}{dt^2} + \{^{\,i}_{jk}\} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{4.43}$$

This part of the information contained in the system (4.40) is therefore n-symplectic gauge-invariant and agrees with the corresponding result from the cotangent bundle. That this must be so can be seen as follows. The structure equation for $\hat{g}^{ij}$ is

$$d\hat{g}^{ij} = -2 X_{\hat{g}}{}^{(i} \lrcorner \, d\theta^{j)} \tag{4.44}$$

Setting $i = j = 0$ we obtain an equation for $X_{\hat{g}}{}^0$, namely

$$d\hat{g}^{00} = -X_{\hat{g}}{}^0 \lrcorner\, d\theta^0 \tag{4.45}$$

which is essentially the LM form of Hamiltonian's equations on the cotangent bundle, based on the "kinetic energy Hamiltonian" $\hat{g}^{00} = g^{ij}(x)\pi_i^0\pi_j^0$, with π_i^0 playing the role of the momentum coordinate p_i.

Next consider the remaining equations (4.42) which determines the transport of the spatial coframe along the geodesic, and which we will refer to as the *triad transport law*. Substituting $\pi_a^0 = g_{ab}\dot{x}^b$ from the first of equations (4.41), and again using the relationship between the connection coefficients, the Christoffel symbols of the metric tensor, and the non-metricity, one can reduce this equation to the form

$$\frac{D\pi_a^\alpha}{Ds} = \frac{1}{2}\left(Q_{k\cdot a}^{\cdot l\cdot} - Q^l{}_{\cdot ak}\right)\dot{x}^k\pi_l^\alpha \tag{4.46}$$

where the covariant derivative on the left-hand-side is taken with respect to the Levi-Civita connection defined by the metric tensor. We consider two cases:

1. If we choose the Levi-Civita connection for the n-symplectic gauge, then $Q_a{}^{jk} \equiv 0$, and the triad is *parallel* transported along the geodesic. Since the unit tangent is also parallel transported along the geodesic, this case corresponds to the true **free observer** in spacetime, namely a freely falling (trajectory is a non-accelerating geodesic), non-rotating (triad is parallel transported along a geodesic) observer.

2. If we choose for the n-symplectic gauge an arbitrary torsion-free connection that is not the unique Levi-Civita connection of $\vec{g}$, then $Q_a{}^{jk} \neq 0$, and the triad is no longer parallel transported along the geodesic. Since the unit tangent is still parallel transported along the geodesic, this case corresponds to a freely falling (trajectory is a non-accelerating geodesic) observer who three space is subject to the forces given on the right-hand-side of (4.46). These forces includes, but are not restricted to, rotations.

5 Conclusions

In this paper we have shown that the Schouten-Nijenhuis brackets of contravariant tensor fields on a manifold M have a natural and fundamental geometrical interpretation in terms of n-symplectic geometry on the bundle of linear frames LM of the manifold M. From an abstract algebraic point of view one might prefer the axiomatic, base manifold definition of the Schouten-Nijenhuis brackets. For example, for the anti-symmetric contravariant tensor fields we have:

> **Definition:** (See [13, 17]) The Schouten-Nijenhuis bracket of anti-symmetric contravariant tensor fields on a manifold M is the unique $\mathbb{R}$-bilinear mapping
> $[\ ,\ ] : A\mathcal{X}(M) \times A\mathcal{X}(M) \to A\mathcal{X}(M)$, which
>
> 1. extends the Lie bracket of vector fields,
> 2. satisfies $[X, f] = \mathcal{L}_X(f)$ for all vector fields X and all smooth functions f on M,
> 3. is graded antisymmetric,
> 4. is a graded biderivation of $A\mathcal{X}(M)$.

However, this abstract definition does not lend itself easily to geometrical analysis. On the otherhand we have seen that one may define the Schouten-Nijenhuis bracket for anti-symmetric contravariant tensor fields on M geometrically on LM as follows:

> **Definition:** For $f \in A\mathcal{X}^q(M)$, $g \in A\mathcal{X}^r(M)$ define $[f, g]$ to be the unique element of $A\mathcal{X}^{q+r-1}(M)$ determined by
>
> $$\widehat{[f, g]} = \{\hat{f}, \hat{g}\} \qquad (5.47)$$
>
> where the bracket on the right-hand-side is the n-symplectic bracket defined in equation (2.13), and $\hat{f}$ and $\hat{g}$ are the tensorial functions on LM uniquely determined by f and g, respectively.

Since the n-symplectic bracket is defined in (2.14) in terms of the n-symplectic Hamiltonian operators, which themselves are equivalence classes of sets of vector fields, this definition of the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$ is clearly more geometrical, and lends itself to geometrical analysis.

There is a strong parallel between these two definitions of the Schouten-Nijenhuis brackets and two definitions of linear connections on a manifold M, namely the axiomatic Koszul definition on the base manifold M, and the geometrical definition of a linear connection on LM. We recall that in the axiomatic approach one defines a linear connection on M as an operator ∇ with a certain set of properties, much like the first definition given above for the Schouten-Nijenhuis bracket $[\ ,\ ]_{S/N}$. On the otherhand, just as we have defined the Schouten-Nijenhuis brackets on LM one may define [8] a linear connection geometrically on LM as a horizontal distribution (or, equivalently, as a Lie-algebra-valued connection 1-form). The axiomatic and frame bundle definitions of linear connections are equivalent, as are the two definitions of the Schouten-Nijenhuis brackets given above. However, it is clear that for geometrical analysis, the LM version of the definition of a linear connection for M is the superior definition. The frame bundle version of linear connections is also clearly more fundamental, as it is the approach taken in *foundational studies* of differential geometry [4]. It is our assertion that the frame bundle version of the Schouten-Nijenhuis brackets is also more geometrical and fundamental than the base manifold version of the Schouten-Nijenhuis brackets. We have already demonstrated in Sections 3 and 4 the geometrical utility of the n-symplectic version of the Schouten-Nijenhuis brackets. That the frame bundle version of the Schouten-Nijenhuis brackets is also more fundamental can be demonstrated by pursuing the above analogy a little further.

We recall that a fundamental feature of the frame bundle definition of a linear connection is that once such a connection is specified, one may use that connection to define covariant differentiation of tensor fields on M in terms of the exterior covariant

[4] See, for example, [8, 1, 11]

differentiation of the associated tensorial fields on LM. In the construction one makes essential use of the fact that the various tensor bundles may be considered as vector bundles associated to LM. Alternatively, one may use the connection on LM to induce a connection for covariant differentiation of sections of the tensor bundles [8], thereby recovering the axiomatic Koszul definition of a connection. The point to be stressed here is that all features of linear connections on the tensor bundles flow from the basic, unifying definition of a linear connection on LM.

We point out that the n-symplectic structure on LM also plays a basic, unifying role for the various symplectic-type theories on the appropriate tensor bundles. In [16] it was argued that the canonical 1-form ϑ on the cotangent bundle of a manifold M is induced from the n-symplectic structure on LM, and that in fact all features of the symplectic geometry of tensorial observables on T^*M are induced from corresponding structures on LM. Hence the symplectic geometry for classical particle mechanics is induced from the n-symplectic geometry on LM. More recently [4] it was pointed out that the n-symplectic structure on LM induces a "canonical p-form" on each of the form bundles $\Lambda_p M$, $1 \leq p \leq n$. As a special case one may suppose that $M \to N$ is itself a vector bundle over a manifold N, with $\dim(N)=k$. The previous theorem asserts [4, 12] that the n-symplectic structure on LM induces a canonical k-form on the bundle of k-forms $\Lambda_k M$. It has been shown [6, 7] that a certain subbundle Z of $\Lambda_k M$ is isomorphic with the bundle of affine cojets of sections of $M \to N$, and that this subbundle Z is the appropriate phase space for a field theory in which the sections of $M \to N$ are the fields of the theory. In particular, Z supports a canonical k-form, the so-called "multisymplectic k-form". In [4, 12] it is shown that this multisymplectic form is in fact the restriction to the subbundle Z of the form on $\Lambda_k M$ induced by the n-symplectic structure on LM. Hence n-symplectic geometry on LM not only induces the canonical symplectic 1-form for classical particle mechanics, but it also induces the canonical multisymplectic k-form for classical field theory. Just as the basic notions about covariant differentiation of tensor fields can be traced back to, and are induced by, a linear connection defined on LM, the basic features of symplectic and multisymplectic geometry on the form bundles can be traced back to, and are induced by, the n-symplectic geometry on LM. These facts clearly establish the fundamental nature of n-symplectic geometry on frame bundles.

Appendix: Some facts about n-symplectic geometry

The principal fiber bundle $\pi : LM \longrightarrow M$ of linear frames of an n-dimensional manifold M is the set of pairs (m, e_i) where (e_i) , $i = 1, 2, \ldots, n$ is a linear frame at $m \in M$. The dimension of LM is the even number $n(n+1)$, and the general linear group $GL(n)$ acts on LM on the right by

$$(m, e_i) \cdot g = (m, e_i g^i_j) \tag{A.1}$$

for each $g = (g^i_j) \in GL(n)$. Let (x^i) be a coordinate chart on $U \subset M$. Define coordinates (x^i, π^j_k) on $\hat{U} = \pi^{-1}(U) \subset LM$ by

$$\begin{aligned} x^i(m, e_i) &= x^i(m) \\ \pi^j_k(m, e_i) &= e^j(\tfrac{\partial}{\partial x^k}|_m) \end{aligned} \tag{A.2}$$

where (e^i) denotes the coframe dual to (e_i). Moreover in (A.2) I follow standard conventions and write x^i in place of $x^i \circ \pi$.

Let (r_i), $i = 1, 2, \ldots, n$, denote the standard basis of $\mathbb{R}^n$. Then the $\mathbb{R}^n$-valued soldering one-form $\theta = \theta^i r_i$ on LM may be defined by

$$\theta(X_u) = u^{-1}(d\pi(X_u)) \quad , \quad X_u \in T_u LM \tag{A.3}$$

where $u = (m, e_i) \in LM$ is viewed as the non-singular linear map $u : \mathbb{R}^n \to T_{\pi(u)}M$ given by $u(\xi^i r_i) = \xi^i e_i$. In the local coordinates (x^i, π^j_k) the soldering 1-form θ take the form

$$\theta = (\pi^i_j \, dx^j) r_i \quad . \tag{A.4}$$

Because this form is so similar to the form $\vartheta = p_j\,dx^j$ for the canonical 1-form on T^*M in local canonical coordinates, we refer to the coordinates (x^i, π^j_k) as canonical coordinates on LM. It is not difficult to show that the vector-valued 2-form $d\theta$ is non-degenerate in the sense that

$$X \rfloor d\theta = 0 \iff X = 0 \tag{A.5}$$

An element $\hat{f} \in ST^q(LM)$ determines [15] an equivalence classes $[\![X_{\hat{f}}]\!]$ of $\binom{n+q-2}{q-1}$ vector fields $[\![X_{\hat{f}}^{i_1 \ldots i_{q-1}}]\!]$ via the n-symplectic structure equation

$$d\hat{f}^{i_1 \ldots I_q} = -q! X_{\hat{f}}^{(i_1 \ldots i_{q-1}} \rfloor d\theta^{I_q)} \tag{A.6}$$

where round brackets on indices denotes symmetrization. Note that although $d\theta$ is nondegenerate in the sense of (A.5), because of the symmetrization in (A.6) the non-degeneracy is lost. For a given $\hat{f} \in ST^q$ equation (A.6) only determines the vector fields $X_{\hat{f}}^{i_1 \ldots i_{q-1}}$ up to addition of vector fields $Y^{i_1 \ldots i_{q-1}}$ satisfying the kernel equation

$$Y^{(i_1 \ldots i_{q-1}} \rfloor d\theta^{I_q)} = 0 \quad . \tag{A.7}$$

If a set of vector fields $Y^{i_1 \ldots i_{q-1}}$ satisfies (A.7) then each vector field $Y^{i_1 \ldots i_{q-1}}$ must be vertical. For a given $\hat{f} \in ST^q$ equation (A.6) thus determines an equivalence class of $(\otimes_s)^{q-1}\mathbf{R}^n$-valued Hamiltonian vector fields $([\![X_{\hat{f}}^{i_1 \ldots i_{q-1}}]\!])$, where two $(\otimes_s)^{q-1}\mathbf{R}^n$-valued vector fields are equivalent if their difference satisfies equation (A.7).

An element $\hat{f} = (\hat{f}^{i_1 i_2 \ldots I_q}) \in ST^q$ has the local canonical coordinate representation

$$\hat{f}^{i_1 i_2 \ldots I_q} = f^{j_1 j_2 \ldots j_q}(x) \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{I_q}_{j_q} \quad . \tag{A.8}$$

The associated equivalence classes of Hamiltonian vector fields $[\![X_{\hat{f}}^{i_1 i_2 \ldots i_{q-1}}]\!]$ determined by equation (A.6) have the local coordinate representations

$$\begin{aligned} X_{\hat{f}}^{i_1 i_2 \ldots i_{q-1}} = \Big(\tfrac{1}{(q-1)!} \quad & f^{j_1 j_2 \ldots j_{q-1} k}(x) \tfrac{\partial}{\partial x^k} \\ & - \tfrac{1}{q!} \left(\tfrac{\partial f^{j_1 j_2 \ldots j_q}}{\partial x^a} \pi^b_{j_q} + T_a^{j_1 j_2 \ldots i_{j-1} b} \right) \tfrac{\partial}{\partial \pi^b_a} \Big) \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_{q-1}}_{j_{q-1}} \end{aligned} \tag{A.9}$$

where the functions $T_a^{i_1 i_2 \ldots i_{q-1} b}$ must satisfy

$$T_a^{(i_1 i_2 \ldots i_{q-1} b)} = 0 \tag{A.10}$$

but are otherwise arbitrary. These functions $T_a^{i_1 i_2 \ldots i_{q-1} b}$ thus represent the undetermined part of $[\![X_{\hat{f}}]\!]$ for $f \in ST^q(LM)$.

An element $\hat{f} \in AT^q(LM)$ determines [15] an equivalence classes $[\![X_{\hat{f}}]\!]$ of $\binom{n}{q-1}$ vector fields $[\![X_{\hat{f}}^{i_1 \ldots i_{q-1}}]\!]$ via the n-symplectic structure equation

$$d\hat{f}^{i_1 \ldots I_q} = -q! X_{\hat{f}}^{[i_1 \ldots i_{q-1}} \rfloor d\theta^{I_q]} \tag{A.11}$$

where square brackets [ ] on indices denotes anti-symmetrization. The anti-symmetrization in (A.11) again introduces a degeneracy in the determination of the Hamiltonian operators.. For a given $\hat{f} \in AT^q$ equation (A.11) only determines the vector fields $X_{\hat{f}}^{i_1 \ldots i_{q-1}}$ up to addition of vector fields $Y^{i_1 \cdots i_{q-1}}$ satisfying the anti-symmetric kernel equation

$$Y^{[i_1 \ldots i_{q-1}} \rfloor d\theta^{I_q]} = 0 \quad . \tag{A.12}$$

If a set of vector fields $Y^{i_1 \cdots i_{q-1}}$ satisfies (A.12) then each vector field $Y^{i_1 \cdots i_{q-1}}$ must be vertical. For a given $\hat{f} \in AT^q$ equation (A.11) thus determines an equivalence class of $(\otimes_s)^{q-1}\mathbf{R}^n$-valued Hamiltonian vector fields $([\![X_{\hat{f}}^{i_1 \ldots i_{q-1}}]\!])$, where two $(\otimes_s)^{q-1}\mathbf{R}^n$-valued vector fields are equivalent if their difference satisfies equation (A.12).

An element $\hat{f} = (\hat{f}^{i_1 i_2 \ldots I_q}) \in ST^q$ has the local canonical coordinate representation

$$\hat{f}^{i_1 i_2 \ldots I_q} = f^{j_1 j_2 \ldots j_q}(x) \pi_{j_1}^{i_1} \pi_{j_2}^{i_2} \cdots \pi_{j_q}^{I_q} \quad . \tag{A.13}$$

The associated equivalence classes of Hamiltonian vector fields $[\![X_{\hat{f}}^{i_1 i_2 \ldots i_{q-1}}]\!]$ determined by equation (A.11) have the local coordinate representations

$$X_{\hat{f}}^{i_1 i_2 \ldots i_{q-1}} = \Big(\tfrac{1}{(q-1)!} \quad f^{j_1 j_2 \ldots j_{q-1} k}(x) \tfrac{\partial}{\partial x^k} - \tfrac{1}{q!} \left(\tfrac{\partial f^{j_1 j_2 \cdots j_q}}{\partial x^a} \pi_{j_q}^b + T_a^{j_1 j_2 \ldots i_{j-1} b} \right) \tfrac{\partial}{\partial \pi_a^b} \Big) \pi_{j_1}^{i_1} \pi_{j_2}^{i_2} \cdots \pi_{j_{q-1}}^{i_{q-1}} \tag{A.14}$$

where the functions $T_a^{i_1 i_2 \ldots i_{q-1} b}$ must now satisfy

$$T_a^{[i_1 i_2 \ldots i_{q-1} b]} = 0 \tag{A.15}$$

but are otherwise arbitrary. These functions $T_a^{i_1 i_2 \dots i_{q-1} b}$ thus represent the undetermined part of $[\![X_{\hat{f}}]\!]$ for $f \in ST^q(LM)$.

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