Einstein-Maxwell dynamics as a P(4) affine theory

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In Newtonian mechanics the energy of a particle is defined only up to an arbitrary additive constant. By using affine functions to model the Newtonian energy we show that it is possible to reformulate arbitrary time- and velocity-independent forces as R^1 affine gauge potentials. Solutions of Newton's second law then define R^1 affine energy geodesics, and the R^1 flat gauge potentials are shown to correspond to conservative Newtonian forces. We generalize these ideas to relativistic mechanics by modeling the energy-momentum of classical particles as R^4 affine four-vectors. If this R^4 affine structure is to be compatible with the O(1,3) Riemannian structure of spacetime, then the R^4 gauge potential must correspond to an antisymmetric tensor field on spacetime, and this field is identified with the electromagnetic field tensor. We are eventually led to a reformulation of the Einstein-Maxwell theory as a $P(4)=O(1,3)\otimes R^4$ affine gauge theory in which the timelike affine geodesics correspond to Lorentz-force-law trajectories, and the Einstein-Maxwell field equations are reformulated as gauge field equations in terms of the P(4) curvature.

I. INTRODUCTION

A unified theory of gravitation and electromagnetism based on the Poincaré group $P(4)=O(1,3)\otimes R^4$ is presented in this paper. In the theory the subgroup R^4 arises as the gauge group for affine-vector fields that are used to model the energy-momentum of classical charged particles on spacetime. The affine-vector model is introduced in order to deal with a certain arbitrariness that is implicit in the definition of the energy-momentum for classical particles, and it may be considered as a generalization of the situation in Newtonian mechanics where the total energy of a particle moving under the influence of a conservative force[1] $\mathbf{F}$ is defined only up to an arbitrary additive constant. In an inertial frame the total energy is $E=\frac{1}{2}mv^2+V$ and the potential energy function V, defined by

$$V(p)=-\int_{p_0}^{p}\mathbf{F}\cdot d\mathbf{l}\ , \tag{1.1}$$

clearly depends on the reference point p_0. Upon changing the reference point from p_0 to p_1 the potential $V(p)$ changes to $V(p)+\delta V(p_0,p_1)$, where

$$\delta V(p_0,p_1)=\int_{p_0}^{p_1}\mathbf{F}\cdot d\mathbf{l}=\text{const}\ . \tag{1.2}$$

The total energy of a particle defined relative to p_0 and p_1 also differs by the constant $\delta V(p_0,p_1)$, and since any point may be chosen as the reference point, the additive constant is completely arbitrary.

A given conservative force thus does not define a unique potential energy function, but rather it defines a class of potential energy functions all differing one from another by a constant. It is customary to ignore this fact, mainly because it causes no problems with the law of conservation of energy, $\frac{1}{2}mv^2+V$ being constant with respect to all reference points if it is constant with respect to one.

In this paper this arbitrariness in the definition of the energy of a Newtonian particle is reexamined, developed, and then generalized to relativistic mechanics. The fundamental idea in the Newtonian case is to incorporate the arbitrary additive constant into the definition of the energy of a particle, and this will be done by modeling the energy as an affine function instead of a real-valued function. The generalization to relativistic mechanics leads to an affine-vector model for the energy-momentum of a particle, and when this affine-vector model is properly formulated on the spacetime manifold we find that the natural R^4 gauge potential defined by the new affine degrees of freedom corresponds to a second-rank antisymmetric tensor field which we identify with the electromagnetic field tensor. We are eventually led to a reformulation of the Einstein-Maxwell theory as a P(4) $=O(1,3)\otimes R^4$ affine gauge theory in which the timelike affine geodesics correspond to Lorentz-force-law trajectories, and the Einstein-Maxwell field equations are reformulated as gauge-field equations in terms of the P(4) curvature.

In order to see how affine geometry may be used to model an additive arbitrariness in a physical theory, we begin in Sec. II by reformulating the definition of the energy of Newtonian particles in terms of affine functions. An arbitrary time- and velocity-independent force $\mathbf{F}=F^i\partial_i$ is interpreted as defining an R^1-gauge potential $\mathbf{K}=F_i dx^i$, the R^1-translational degree of freedom corresponding to the additive arbitrariness in the definition of the energy. After introducing the concept of R^1 affine energy geodesics, we show that such curves are defined by solutions of Newton's second law. The R^1 curvature corresponding to the potential $\mathbf{K}$ is defined, and we show that the flat (integrable) R^1-gauge potentials correspond to conservative Newtonian forces.

In Sec. III these ideas are generalized to relativistic mechanics by modeling the energy-momentum of classical charged particles as affine vectors on spacetime. We first consider covariant constant electromagnetic fields on Minkowski spacetime and show that for such fields one can define an integrable electromagnetic potential energy-momentum affine-vector field $q\theta^j$ such that $\pi^j \equiv mu^j + q\theta^j$ is constant along a trajectory that is a solution of the Lorentz force law for charged particles. The electromagnetic potential energy-momentum $q\theta^j$ is defined only up to an additive constant four-vector because it depends on a reference point in a manner analogous to the dependence of the Newtonian potential on reference point, and this leads to the affine-vector model for the energy-momentum of classical charged particles.

In order to deal with more general fields we next introduce an initially arbitrary R^4-gauge potential $\mathbf{K} = K^i{}_j \partial_i \otimes dx^j$, the R^4-translational degrees of freedom corresponding now to the additive arbitrariness in the energy-momentum of a particle. After generalizing the R^1-affine energy geodesics introduced in Sec. II to R^4-affine energy-momentum geodesics, we show that in a certain zero-translation gauge in which the linear part of the energy-momentum is the kinetic energy-momentum per unit mass $p^j = u^j$, the tensor field $K_{ij} dx^i \otimes dx^j$ must be antisymmetric, and we identify this tensor field with the electromagnetic field tensor. The timelike affine geodesics of type σ are then solutions of the Lorentz force law for charged particles with charge-to-mass ratio σ. In order to identify this zero-translation gauge physically we examine how charged particles are observed relative to uncharged particles, and we show in Sec. IV that the zero-translation gauge is canonically defined by uncharged particles. In the development the R^4 curvature is introduced, and the integrable R^4-gauge potentials are shown to describe covariant constant electromagnetic fields on spacetime.

The physical ideas presented in Secs. III and IV are made rigorous in Secs. V–VII. In Sec. V an Einstein-Maxwell spacetime is reinterpreted as an Einstein-Maxwell affine spacetime, and charged particle trajectories are then geometrized as affine geodesics. In Sec. VI the Maxwell equations are geometrized in terms of the R^4 curvature, and the geometrization is completed in Sec. VII where we show that the Einstein-Maxwell field equations can be reformulated as geometrical equations in terms of the P(4) curvature. A summary and conclusions are given in Sec. VIII.

II. NEWTONIAN ENERGY AFFINE FUNCTIONS

Let a classical particle of mass m move along a curve $\mathbf{r}(t)$ in an inertial frame under the influence of a conservative force $\mathbf{F}$. Newton's second law of motion $(d/dt)(m\mathbf{v}) = \mathbf{F}$ leads to the energy equation

$$\frac{d}{dt}(\tfrac{1}{2}mv^2) - \mathbf{v}\cdot\mathbf{F} = 0 , \tag{2.1}$$

and upon using the definition (1.1) for any reference point Eq. (2.1) can be rewritten as

$$\frac{d}{dt}(\tfrac{1}{2}mv^2 + V) = 0 . \tag{2.2}$$

The integrated form of this law of conservation of energy states that for any two points p_1 and p_2 along the trajectory

$$(\tfrac{1}{2}mv^2 + V)(p_2) - (\tfrac{1}{2}mv^2 + V)(p_1) = 0 , \tag{2.3}$$

and it is customary to say that the total energy defined by

$$E \equiv \tfrac{1}{2}mv^2 + V \tag{2.4}$$

is thus a constant of the motion.

Note that the quantity $\frac{1}{2}mv^2 + V$ is used in two different ways in (2.3) and (2.4), since only its difference occurs in (2.3) while it is used to define the quantity E in (2.4). That is to say, while the law of conservation of energy only refers to the difference between total energies at two points, Eq. (2.4) gives an independent existence to the total energy at each point. This shows that any constant can be added to the right-hand side of the definition (2.4) without affecting the law of conservation of energy in the form

$$E(p_2) - E(p_1) = 0 , \tag{2.5}$$

and this arbitrariness can be traced to the definition (1.1) of the potential as discussed earlier.

We shall model this arbitrariness in the definition of the energy at each point in space as follows. We assign to each point p in Newtonian space a local energy space $\hat{\mathscr{E}}_p$. These local spaces are to have the property that the difference between two energies at any one point is well defined, but no natural zero exists for each space so that individual energies are known only up to an arbitrary additive constant. Mathematically the $\hat{\mathscr{E}}_p$ are one-dimensional affine spaces,[2] the difference between two affine energies $\hat{E}_1$ and $\hat{E}_2$ at p being given by the real number $\delta(\hat{E}_1, \hat{E}_2)$, where $\delta: \hat{\mathscr{E}}_p \times \hat{\mathscr{E}}_p \to R^1$ is the difference function. If an arbitrary reference energy, say, $\hat{E}_1(p)$, is selected at each point p, then all other energies at p may be decomposed as

$$\hat{E}(p) = {}^1E(p) \oplus \hat{E}_1(p) . \tag{2.6}$$

The relative energy ${}^1E(p)$ is defined by

$${}^1E(p) = \delta(\hat{E}(p), \hat{E}_1(p)) , \tag{2.7}$$

and it can be thought of as the linear (or vector) component of $\hat{E}(p)$ with respect to the origin field $\hat{E}_1(p)$.

To make these spaces physical we postulate the existence of a vacuum affine energy field $\hat{V}$ such that $\hat{V}(p) \in \hat{\mathscr{E}}_p$ for each p. For the energy of a particle we now write

$$\hat{E}(p) = {}^VE(p) \oplus \hat{V}(p), \quad p = \mathbf{r}(t) . \tag{2.8}$$

Moreover, we assume

$${}^VE(p) = \tfrac{1}{2}mv^2 \equiv T . \tag{2.9}$$

Thus $\hat{V}(p)$ will be chosen as the reference energy at each p so that the physical zero of relative energy at p is assigned to the vacuum.

Now since each local energy space $\hat{\mathscr{E}}_p$ has an independent existence, an energy transport law must be specified in order to correlate information gathered at different points, and this law must be a transport law between affine spaces. For this transport law we take[3]

$$D\hat{V} = -F_i dx^i \tag{2.10}$$

whether or not the force **F** is conservative.

In this affine formulation the R^1-valued one-form $\mathbf{K} = -F_i dx^i$ obtained from the force $\mathbf{F} = F^i \partial_i$ is interpreted as an R^1-gauge potential. The affine covariant derivative of the energy $\hat{E} = T \oplus \hat{V}$ of a particle along its trajectory is[4]

$$D\hat{E}/Dt = dT/dt + D\hat{V}/Dt \ , \tag{2.11}$$

and upon using (2.10) in (2.11) we obtain

$$D\hat{E}/Dt = dT/dt - F_j(dx^j/dt) \ . \tag{2.12}$$

These ideas suggest the following definitions.

Definition 2.1. A classical Newtonian particle[5] is a pair $(m, \mathbf{r}(t))$, where $m \in [0, \infty)$ is the mass of the particle and $\mathbf{r}(t)$ is a curve in R^3, the trajectory of the particle.

Definition 2.2. The energy of a classical Newtonian particle is the one-dimensional affine vector $\hat{E} = T(m, \mathbf{r}(t)) \oplus \hat{V}(\mathbf{r}(t))$ defined along the trajectory of the particle, where $T(m, \mathbf{r}(t)) \equiv \frac{1}{2}mv^2$.

Definition 2.3. The trajectory $\mathbf{r}(t)$ of a classical particle $(m, \mathbf{r}(t))$ is an R^1-affine energy geodesic if its affine energy is parallel along its trajectory, that is, if $D\hat{E}/Dt = 0$ along $\mathbf{r}(t)$.

From Eq. (2.12) and definition 2.3 we have that the trajectory of a classical particle is an affine energy geodesic if

$$D\hat{E}/Dt = \frac{d}{dt}(\tfrac{1}{2}mv^2) - \mathbf{v}\cdot\mathbf{F} = 0 \ . \tag{2.13}$$

This result proves the following theorem.

Theorem 2.1. The trajectory of a classical particle moving under the influence of a force **F** is an affine energy geodesic of the R^1-gauge potential $\mathbf{K} = -F_j dx^j$.

Note that this theorem does not require the force **F** to be conservative.

Return now to the energy transport law (2.10). This law can be used to consistently transport an arbitrarily chosen zero of energy at p to a neighborhood of p if and only if the loop integral $\oint \mathbf{F}\cdot d\mathbf{l}$ vanishes for all loops at p. An application of Stokes's theorem shows that this requires $\nabla \times \mathbf{F} = 0$, which is of course the usual condition for a conservative force. In the affine theory this condition arises in a different but equivalent way as follows.

Since the group R^1 is commutative, the curvature of the R^1-gauge potential $\mathbf{K} = -F_i dx^i$ is the R^1-valued two-form[3]

$$\Phi = \Phi_{ij} dx^i \wedge dx^j = -\partial_{[i} F_{j]} dx^i \wedge dx^j \ .$$

The R^1-gauge curvature is flat (i.e., integrable in the sense described above) if and only if $\Phi = 0$. This implies $\partial_{[i} F_{j]} = 0$, which is equivalent to $\nabla \times \mathbf{F} = 0$. We have proved the following theorem.

Theorem 2.2. The R^1-affine gauge potential $\mathbf{K} = -F_j dx^j$ defined by a time- and velocity-independent force $\mathbf{F} = F^j \partial_j$ is integrable if and only if **F** is conservative.

The above results are of course mainly a reformulation of standard Newtonian concepts. However, they show that a consistent reformulation of Newtonian forces as R^1-affine energy potentials is possible. Moreover, in the reformulation the Newtonian energy equation (2.1) occurs as the equation of an affine energy geodesic.

Theorem 2.1 shows that solutions of $\mathbf{F} = m\mathbf{a}$ for time- and velocity-independent forces are affine energy geodesics. Time-dependent forces will be dealt with implicitly in the next section on relativistic mechanics, but here we wish to briefly consider if there are solutions of $\mathbf{F} = m\mathbf{a}$ for velocity-dependent forces that are also affine energy geodesics. At least one special case is clear, namely, $\mathbf{F}_1 = \mathbf{v} \times \mathbf{B}$ for some vector field **B**. In this case $\mathbf{v}\cdot\mathbf{F}_1 = 0$ and if $m\mathbf{a} = \mathbf{F}_1 + \mathbf{F}_2$ with $\mathbf{F}_2$ conservative, then solutions of this equation will also be energy geodesics of the R^1 potential defined by $\mathbf{F}_2$ alone. However, it does not seem possible to use $\mathbf{v} \times \mathbf{B}$ to define an R^1 gauge potential. In fact, since such forces "do no work" and thus do not contribute to the energy they should not fit into the R^1 formalism. However, one might speculate that the three-momentum $\mathbf{p} = m\mathbf{v}$ could be treated as an affine vector, and then the force $\mathbf{v} \times \mathbf{B}$ could be used to define an R^3-gauge potential. This can be done,[6] but we prefer to treat this aspect of the problem relativistically in the next section.

One further remark is in order. Above we have tacitly assumed that all classical particles couple to the given Newtonian field, and it was this assumption that led to the postulate of a vacuum affine energy field $\hat{V}$. As we shall see in Sec. IV, if certain particles do not couple to a given field, then these "uncharged" particles may be used to define the reference energy field.

III. RELATIVISTIC ENERGY-MOMENTUM AFFINE VECTORS

In relativistic mechanics the separate Newtonian concept of energy and three-momentum of a particle are not Lorentz invariant, and they must be replaced by the Lorentz-invariant energy-momentum four-vector. In this section we generalize the above reformulation of Newtonian energy as an affine function to relativistic mechanics by modeling the energy-momentum of classical particles as affine four-vectors. Rather than considering general forces we will initially consider only the classical electromagnetic interaction, and in order to see how an affine-vector model can arise for the energy-momentum of a classical charged particle we first consider static, uniform electromagnetic fields in Minkowski spacetime M.

The equations of motion of a classical charged particle of mass m and charge q in flat spacetime are

$$\frac{d}{ds}(mu^j) = qF^j{}_k(dx^k/ds) \ . \tag{3.1}$$

Here s is the proper time, $u^j = dx^j/ds$ is the four-velocity of the particle, and the $F^j{}_k$ are the components of the electromagnetic field tensor in global Lorentzian coordinates.

When the electromagnetic field is static and uniform $\partial_i F^j{}_k = 0$, and using this fact, Eq. (3.1) can be rewritten as

$$\frac{d}{ds}(mu^j - qF^j{}_k x^k) = 0 \ . \tag{3.2}$$

We can conclude from this equation that when the electromagnetic field is covariant constant in flat spacetime the quantity

$$\pi^j \equiv mu^j - qF^j{}_k x^k \tag{3.3}$$

is constant along the trajectory of the charged particle. We may thus call π^j the *total energy-momentum* of the particle. It is the sum of the *kinetic energy-momentum* $p^j = mu^j$ and the electromagnetic *potential energy-momentum*[7] $q\theta^j = -qF^j{}_k x^k$. Thus a charged particle in a covariant constant electromagnetic field in Minkowski spacetime will move along a world line so as to keep the sum of its kinetic and potential energy-momenta constant.

The electromagnetic potential energy-momentum four-vector θ^j can be defined for a given covariant constant electromagnetic field by

$$\theta^j(p) = -\int_{p_0}^{p} F^j{}_k dx^k$$
$$= -F^j{}_k[x^k(p) - x^k(p_0)] \ . \tag{3.4}$$

This definition is a relativistic analog of the Newtonian definition (1.1) and θ^j, like the Newtonian potential, clearly depends on the reference point p_0. Upon changing the reference point from p_0 to p_1 the potential θ^j changes from θ^j to $\theta^j + \delta\theta^j(p_0,p_1)$ where

$$\delta\theta^j(p_0,p_1) = \int_{p_0}^{p_1} F^j{}_k dx^k$$
$$= F^j{}_k[x^k(p_1) - x^k(p_0)] = \text{const} \ . \tag{3.5}$$

Thus a covariant constant electromagnetic field on Minkowski spacetime does not define a unique potential energy-momentum four-vector, but rather a class of four-vectors each differing one from another by a constant four-vector. As a result the total energy-momentum (3.3) of a charged particle in such a field is also only known up to an additive constant four-vector. Any constant four-vector may thus be added to the right-hand side of the definition (3.3) without affecting the law of conservation of energy-momentum in the form

$$\pi^j(p_2) - \pi^j(p_1) = 0 \ . \tag{3.6}$$

The arbitrariness in the definition (3.3) of the total energy-momentum of a classical charged particle is clearly analogous to the arbitrariness in the definition of the Newtonian total energy, and an obvious analog of the local energy affine spaces $\hat{\mathscr{E}}_p$ is suggested. We assign to each event p in spacetime a local energy-momentum space $\hat{\Pi}_p$. Each space $\hat{\Pi}_p$ is a four-dimensional affine space[2] whose elements are energy-momentum affine vectors $\hat{\pi}$. No absolute zero exists in an affine space so that individual energy-momenta in $\hat{\Pi}_p$ are known only up to an additive four-vector at each event p. The difference between two affine energy-momenta $\hat{\pi}_1$ and $\hat{\pi}_2$ at p is the four-vector $\delta(\hat{\pi}_1,\hat{\pi}_2)$, where $\delta:\hat{\Pi}_p \times \hat{\Pi}_p \to T_pM$ is the difference function.

If a reference energy-momentum affine-vector field $\hat{\pi}_1$ is chosen arbitrarily, then at each $p \in M$ all other energy-momenta in $\hat{\Pi}_p$ can be decomposed as

$$\hat{\pi}(p) = {}^1\hat{\pi}(p) \oplus \hat{\pi}_1(p) \ . \tag{3.7}$$

The four-vector ${}^1\hat{\pi}$ is the linear component of $\hat{\pi}$ with respect to the origin field $\hat{\pi}_1$, and it is defined by

$${}^1\hat{\pi}(p) = \delta[\hat{\pi}(p), \hat{\pi}_1(p)] \ . \tag{3.8}$$

We will assume this affine structure for the local energy-momentum spaces $\hat{\Pi}_p$ whether or not the electromagnetic field is covariant constant. More generally, we shall temporarily drop specific reference to electromagnetic fields and consider instead a general affine transport law between the affine spaces $\hat{\Pi}_p$. On flat Minkowski spacetime M such a transport law is specified by an affine covariant derivative[3]

$$D\hat{\theta}^j = \sigma\{{}^\theta K^j{}_k dx^k\} \ . \tag{3.9}$$

In this equation D denotes the affine covariant derivative operator, σ is a coupling constant, and the type (1,1) tensor ${}^\theta K^j{}_k \partial_j \otimes dx^k$ is the tensorial component of the R^4 connection $\hat{\mathbf{K}}$ in the translation gauge $\hat{\theta}$. As connection coefficients the ${}^\theta\mathbf{K}$ are thought of as four-vector-valued one-forms, but each ${}^\theta\mathbf{K}$ is a type (1,1) tensor field on spacetime. If we choose $\hat{\Sigma}$ as origin field instead of $\hat{\theta}$ then we have

$$D\hat{\Sigma}^j = \sigma\{{}^\Sigma K^j{}_k dx^k\} \ . \tag{3.10}$$

The type (1,1) tensor fields ${}^\theta K^j{}_k$ and ${}^\Sigma K^j{}_k$ are related by

$${}^\Sigma K^j{}_k = {}^\theta K^j{}_k + \partial_k t^j \ , \tag{3.11}$$

where $\sigma\mathbf{t} = {}^\theta\boldsymbol{\Sigma} = \delta(\hat{\boldsymbol{\Sigma}}, \hat{\boldsymbol{\theta}})$. Thus

$$\hat{\boldsymbol{\Sigma}}(p) = \sigma\mathbf{t}(p) \oplus \hat{\boldsymbol{\theta}}(p) \quad \text{for each } p \in M \ . \tag{3.12}$$

Equation (3.11) is the affine transformation law[3] for the R^4-connection coefficients when the reference gauge is changed from $\hat{\boldsymbol{\theta}}$ to $\hat{\boldsymbol{\Sigma}}$ as in (3.12).

Finally, suppose $\hat{\pi}$ is an affine-vector field on spacetime, and let us take $\hat{\boldsymbol{\theta}}$ as origin field. Thus

$$\hat{\pi}^j(p) = {}^\theta\pi^j(p) \oplus \hat{\theta}^j(p) \quad \text{for each } p \in M \ . \tag{3.13}$$

Then the affine covariant derivative of $\hat{\pi}^j$ is

$$D_k \hat{\pi}^j = \partial_k({}^\theta\pi^j) + D_k \hat{\theta}^j \ . \tag{3.14}$$

Using (3.9) in this equation yields the formula

$$D_k \hat{\pi}^j = \partial_k({}^\theta\pi^j) + \sigma\{{}^\theta K^j{}_k\} \ . \tag{3.15}$$

The affine covariant derivative of $\hat{\pi}^j$ along a curve $[x^k(s)]$ is thus

$$D\hat{\pi}^j/Ds = d({}^\theta\pi^j)/ds + \sigma\{{}^\theta K^j{}_k(dx^k/ds)\} \ . \tag{3.16}$$

In order to make the local energy-momentum affine spaces physical we introduce the following postulate.

Postulate 1. There exists a reference energy-momentum gauge $\hat{\boldsymbol{\beta}}$ such that ${}^\beta\boldsymbol{\pi} = \delta(\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\beta}}) = \mathbf{u}$ for each classical particle with energy-momentum $\hat{\boldsymbol{\pi}}$ and four-velocity $\mathbf{u}$.

We shall refer to the gauge $\hat{\boldsymbol{\beta}}$ singled out by this postulate as the *zero-translation gauge,* since each particle's kinetic energy-momentum per unit mass is untranslated in this gauge. The physical identification of such a gauge will be discussed in the next section. We have used the kinetic energy-momentum per unit mass in the postulate rather than $m\mathbf{u}$ in order to incorporate into the structure of the theory (cf. Sec. IV) the fact that all classical charged particles with the same charge-to-mass ratio follow identical trajectories in spacetime.

Consider now a particle with timelike world line $x^j(s)$ and affine energy-momentum $\hat{\pi}(s)$ defined along $x^j(s)$. The affine covariant derivative of $\hat{\pi}(s)$ along $x^j(s)$ is by (3.16) and postulate 1,

$$D\hat{\pi}^j/Ds = d(u^j)/ds + \sigma[{}^{\beta}K^j{}_k(dx^k/ds)] \ . \quad (3.17)$$

This equation may be compared with the Newtonian equation (2.12), and proceeding as in Sec. II we introduce the following definitions.

Definition 3.1. A classical particle is a triple $(m,\sigma,x^j(s))$, where $m\in[0,\infty)$ is the mass of the particle, $\sigma\in(-\infty,\infty)$ is the affine charge of the particle, and $x^j(s)$ is a future-pointing timelike $(m>0)$ or null $(m=0)$ world line in spacetime.

Definition 3.2. The energy-momentum of a classical particle is the affine four-vector $\hat{\pi}=\mathbf{u}\oplus\hat{\beta}$ defined along the particle's world line, where $\mathbf{u}$ is the instantaneous four-velocity of the particle.

Definition 3.3. The world line of a classical particle is an R^4-affine energy-momentum geodesic if the energy-momentum of the particle is affine parallel along its world line, that is, if $D\hat{\pi}/Ds=0$ along $x^j(s)$.

From Eq. (3.17) and definition 3.3 we have that the world line of a classical particle is an energy-momentum affine geodesic if

$$du^j/ds + \sigma({}^{\beta}K^j{}_k u^k)=0 \ . \quad (3.18)$$

Up to this point we have not had to place any restrictions on the R^4-affine connection $\hat{\mathbf{K}}$. However, if Eq. (3.18) is to be compatible with the Riemannian structure of spacetime, then the identity $u_j(du^j/ds)=0$ satisfied by the unit tangent vector to a curve and Eq. (3.18) imply

$${}^{\beta}K_{jk}u^j u^k=0 \ .$$

This equation will hold for arbitrary unit tangent vectors if and only if

$${}^{\beta}K_{(jk)}=0 \ .$$

The result is that postulate 1 and the affine geodesic equation will be compatible with the Riemannian structure of spacetime if and only if the R^4-connection $\hat{\mathbf{K}}$ is antisymmetric in the zero-translation gauge, that is, if and only if

$${}^{\beta}K_{jk}={}^{\beta}K_{[jk]} \ .$$

Since this compatibility should be required of a relativistic theory, we introduce the following postulate.

Postulate 2. The R^4-connection $\hat{\mathbf{K}}$ is antisymmetric in the zero-translation gauge.

If we now identify ${}^{\beta}K_{jk}$ with the negative of the electromagnetic field tensor $-F_{jk}$, then the affine geodesic equation (3.18) may be rewritten as

$$du^j/ds - \sigma F^j{}_k u^k=0 \ . \quad (3.19)$$

This equation is the *Lorentz force law for a classical charged particle with charge-to-mass ratio* σ.

To summarize, we have shown that postulates 1 and 2 require ${}^{\beta}\mathbf{K}$ to be an antisymmetric tensor field on spacetime, and if this tensor field is identified with the negative of the electromagnetic field tensor, then the affine energy-momentum geodesic equation is the Lorentz force law for charged particles. Now while postulate 2 is a mathematical compatibility condition, postulate 1 is a physical statement and an operational definition of the zero-translation gauge is needed in order for postulate 1 to be meaningful. In the next section we describe a thought experiment that shows that the usual concept of instantaneous rest frame for charged particles implies a local definition of the zero of energy-momentum that may be used to define the zero-translation gauge.

IV. THE ZERO-TRANSLATION GAUGE

Consider a freely falling laboratory that is free of electromagnetic fields and charges except for a charged test particle at rest in the laboratory. We seek an operational definition of the instantaneous rest frame[8] (IRF) of the charged particle for laboratory times greater than some initial time s_0. As long as the laboratory is freely falling in a region of spacetime in which there is no electromagnetic field, then the laboratory itself can serve as the IRF of the particle. Suppose that at $s_1>s_0$ the laboratory enters a region U of spacetime containing a nonzero electromagnetic field. Then for times s greater than s_1 the laboratory clearly can no longer serve as the IRF since the charged particle will begin to accelerate at time s_1 and hence will not be instantaneously at rest with respect to the laboratory for $s>s_1$.

Thus we need to define the IRF of the charged particle for times $s>s_1$. As a representation of the IRF of the charged particle at $p\in M$ we take an uncharged particle that is unaccelerated and instantaneously comoving with the charged particle at p. This representation of the IRF can be realized as follows.

Let the charged particle enter U at p_1 at time s_1. Then knowing the Maxwell field tensor $F^j{}_k$ in U and the initial energy-momentum $mu^j(s_1)$ of the charged particle at p_1, we can integrate the Lorentz force law

$$\frac{d}{dx}(mu^j)=qF^j{}_k u^k$$

to find the trajectory $x^j(s)$ and $u^j(s)=dx^j/ds$.

Using this solution we define the IRF of the charged particle by

$$(\pi_0)^j(s)=u^j(s) \ , \quad (4.1)$$

$$d_u(\pi_0)^j(s)=0 \ . \quad (4.2)$$

Here the subscript 0 indicates zero charge, and $d_u\pi^j(s)$ denotes the directional derivative of π^j in the direction of u at $x^j(s)$.

The second condition (4.2) distinguishes the charged test particle from the IRF, since the charged particle obeys the set of equations

$$\pi^j(s)=u^j(s) \ , \quad (4.3)$$

$$d_u \pi^j(s) = (q/m) F^j{}_k u^k(s) \ . \tag{4.4}$$

At each $s > s_1$ we have from (4.1) and (4.3) the difference equation

$$\pi(s) - \pi_0(s) = 0_\sigma\{x^j(s)\}, \quad \sigma = q/m \ . \tag{4.5}$$

We interpret this equation as defining an *instantaneous, or local zero of energy-momentum* for the charged particle at each point $x^j(s)$ along its trajectory. Moreover, we *identify* this field of local zeros of energy-momentum with the zero-translation gauge $\hat{\beta}$ assumed in postulate 1.

By analogy with Eq. (4.5) we define $d_u(0_\sigma)^j(s)$ to be the difference between $d_u \pi^j(s)$ and $d_u(\pi_0)^j(s)$ at each point $x^j(s)$:

$$d_u(0_\sigma)^j(s) \equiv d_u \pi^j(s) - d_u(\pi_0)^j(s) \ . \tag{4.6}$$

Using (4.2) and (4.4) in this equation we obtain

$$d_u(0_\sigma)^j = \sigma F^j{}_k u^k \ . \tag{4.7}$$

If we now insist that the result (4.7) be independent of the freely falling laboratory then it must hold for all timelike u^j at each p. We assume that it is true for all vectors u^j at p and generalize (4.7) to

$$d(0_\sigma)^j = \sigma F^j{}_k dx^k \ . \tag{4.8}$$

Equation (4.8) is the differential transport law for the local zero of energy-momentum for charged particles, defined with respect to uncharged particles by Eq. (4.5).

Is this transport law integrable? That is to say, can this law be used to extend the local zero $0_\sigma(p)$ away from p to define a path-independent potential energy-momentum field on a full neighborhood of p? To answer this question we transport $0(p)$ along the two infinitesimal paths $C_1 = pab$ and $C_2 = pcb$ shown in Fig. 1 using the transport law (4.8).

Along the first leg of C_1 we obtain

$$(0_1)^j(a) = 0^j(p) + d_u 0^j(p)$$
$$= 0^j(p) + \sigma F^j{}_k(p) u^k \ .$$

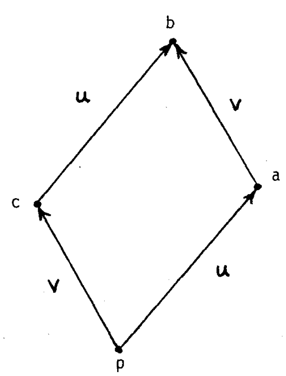

FIG. 1. Two infinitesimal paths, $C_1 = pab$ and $C_2 = pcb$, from p to b.

Going next from a to b in the direction of **v** we obtain

$$(0_1)^j(b) = (0_1)^j(a) + d_v 0^j(a)$$
$$= 0^j(p) + \sigma F^j{}_k(p) u^k + \sigma F^j{}_k(a) v^k \ .$$

Upon expanding $F^j{}_k(a)$ to first order we get

$$(0_1)^j(b) = 0^j(p) + \sigma F^j{}_k(p) u^k + \sigma F^j{}_k(p) v^k$$
$$+ \sigma(\partial_i F^j{}_k)(p) u^i v^k \ .$$

Going from p to b along C_2 we obtain in an analogous manner

$$(0_2)^j(b) = 0^j(p) + \sigma F^j{}_k(p) v^k + \sigma F^j{}_k(p) u^k$$
$$+ \sigma(\partial_i F^j{}_k)(p) v^i u^k \ .$$

The difference $\delta(0_{12})^j(b) := (0_2)^j(b) - (0_1)^j(b)$ between these two values at b is then

$$\delta(0_{12})^j = 2\sigma \partial_{[i} F^j{}_{k]} u^i v^k \ . \tag{4.9}$$

The transport law for the zero of energy-momentum for charged particles will thus be integrable if and only if the right-hand side of (4.9) vanishes for all possible vectors **u** and **v**. Integrability is thus equivalent to ($\sigma = q/m \neq 0$)

$$\Phi_{jk}{}^i \equiv 2\partial_{[j} F^i{}_{k]} = 0 \ . \tag{4.10}$$

If $\Phi_{jk}{}^i \neq 0$ the local zero of energy-momentum for charged particles at p cannot be extended to any neighborhood of p in a path-independent manner using the transport law (4.8). This four-vector-valued two-form $\Phi = \Phi_{jk}{}^i \partial_i \otimes dx^j \wedge dx^k$ is the R^4 curvature of the R^4-affine connection $F^j{}_k \partial_j \otimes dx^k$ (cf. Sec. VI).

In the above discussion we have assumed that the antisymmetric tensor F_{jk} was a Maxwell field tensor, but we have not needed nor have we used the Maxwell equations in the discussion. Let us for the moment simply view F_{jk} as an antisymmetric tensor field and investigate what the integrability condition (4.10) implies about the dynamical behavior of F_{jk}. Upon contracting Eq. (4.10) (set $j = i$) and using the antisymmetry of F_{jk} we obtain

$$\Phi_{jk}{}^j = \partial_j F^j{}_k = 0 \ . \tag{4.11}$$

If we next lower the index i on $\Phi_{jk}{}^i$ and then antisymmetrize over all three indices we obtain from (4.10)

$$\Phi_{[jki]} = 2\partial_{[j} F_{ik]} = 0 \ . \tag{4.12}$$

The integrability condition (4.10) thus requires F_{jk} to satisfy (4.11) and (4.12), which are the *source-free Maxwell equations*. Combining (4.10) and (4.12) one can show that F_{jk} must in fact satisfy

$$\partial_i F_{jk} = 0 \ . \tag{4.13}$$

In summary, if we use the law (4.8) to transport the local definition of the zero of energy-momentum for charged particles, with F_{jk} an arbitrary antisymmetric tensor field on spacetime, then the resulting field will be path-independent if and only if F_{jk} is a covariant constant Maxwell field.

When $\Phi_{jk}{}^i \neq 0$ the transport law (4.8) will not be integrable, whether or not the antisymmetric tensor field F_{jk} satisfies the Maxwell equations. In the nonintegrable case we can ensure that F_{jk} satisfies the source-free Maxwell equations by adopting Eqs. (4.11) and (4.12) as field equations without the stronger condition (4.13) that is implied by (4.10). These equations will be made rigorous in Sec. VI.

V. GEOMETRIZATION OF CLASSICAL PARTICLE TRAJECTORIES

In the preceding sections of this paper we have described the physical basis for an affine theory of the classical interaction of charged particles with the electromagnetic field. In this and the next two sections we put the theory on a firm mathematical foundation by formulating the theory geometrically on the spacetime manifold. The end result will be a P(4) gauge-field description of Einstein-Maxwell dynamics.

In the remainder of this paper we denote an Einstein-Maxwell spacetime by $(M,\mathbf{g},\mathbf{F})$, with $\mathbf{g}$ the metric tensor and $\mathbf{F}$ the Maxwell field tensor on the four-dimensional manifold M. The geometry of an Einstein-Maxwell spacetime is Riemannian and may be regarded as a theory based on an O(1,3) connection. We shall reinterpret such a spacetime as a $P(4)=O(1,3)\otimes R^4$ Poincaré spacetime in the following way.

A generalized affine connection on M can be specified[3] uniquely by a pair $(\Gamma,{}^t\mathbf{K})$, where Γ is a Gl(4) linear connection and ${}^t\mathbf{K}$ is a type (1,1) tensor field representing the R^4 part of the connection in the $\hat{\mathbf{t}}$ translation gauge. Let $(M,\Gamma,{}^t\mathbf{K})$ denote an $A(4)=Gl(4)\times R^4$-affine spacetime based on the affine connection $(\Gamma,{}^t\mathbf{K})$.

Definition 5.1. For each Einstein-Maxwell spacetime $(M,\mathbf{g},\mathbf{F})$ define the associated P(4) *Einstein-Maxwell affine spacetime* to be $(M,\{\ \}_g-\mathbf{F})$, where $\{\ \}_g$ denotes the unique torsion-free linear connection based on $\mathbf{g}$ and ${}^\beta\mathbf{K}=-\mathbf{F}=-F^j{}_k\partial_j\otimes dx^k$ is the Maxwell field tensor in its (1,1) tensor form.

In order to model geometrically the affine energy-momentum geodesics introduced in Sec. III we need the following generalization of a well-known concept in affine differential geometry.[9]

Definition 5.2. For each curve $x^j(s)$ on M and each real number $\sigma\neq 0$ define the σ-tangent affine-vector $\hat{u}^j$ by $\hat{u}^j=dx^j/ds\oplus\hat{\beta}^j$. For $\sigma=0$ define the zero-tangent affine vector to be the usual linear tangent vector to the curve. When $x^j(s)$ is timelike the parameter s is to be chosen as the Riemannian arc length.

The affine-vector field $\hat{\beta}$ in this definition is the zero-translation gauge affine field introduced in Secs. III and IV.

From all possible curves on M we single out those that generalize the concept of linear geodesic.

Definition 5.3. A curve $x^j(s)$ on M is a (generalized) affine geodesic of type σ if its σ-tangent affine vector is parallel along $x^j(s)$. Thus $\hat{u}^j$ satisfies the equation $D_u\hat{u}^j=0$, where $u^j=dx^j/ds$ and D_u denotes the affine covariant derivative operator in the direction of $\mathbf{u}$.

The expanded form of the affine geodesic equation $D_u\hat{u}^j=0$ can be obtained from Eq. (3.18) by replacing du^j/ds by the linear covariant directional derivative $\nabla_u u^j$, so that

$$D_u\hat{u}^j=\nabla_u u^j+\sigma({}^\beta K^j{}_k u^k)=0\ . \tag{5.1}$$

From this equation and the identification ${}^\beta\mathbf{K}=-\mathbf{F}$ in an Einstein-Maxwell affine spacetime we may infer the following theorem.

Theorem 5.1. A timelike affine geodesic of type $\sigma\neq 0$ in an Einstein-Maxwell affine spacetime is the trajectory of a classical charged particle that obeys the Lorentz force law with charge-to-mass ratio σ. A timelike or null affine geodesic of type $\sigma=0$ is the trajectory of a free uncharged particle.

A special feature of the affine theory is that it allows us to incorporate into the formalism the empirical fact that massless particles in nature are always uncharged. If m denotes the mass of a particle, then the particle's electric charge q can be defined by $q=m\sigma$, thus guaranteeing the empirical law that $m=0\Longrightarrow q=0$.

VI. GEOMETRIZATION OF THE MAXWELL EQUATIONS

In a source-free Einstein-Maxwell spacetime the Maxwell field tensor $\mathbf{F}$ satisfies the field equations

$$\nabla_{[i}F_{jk]}=0\ , \tag{6.1}$$

$$\nabla_j F^j{}_k=0\ . \tag{6.2}$$

In the last section we associated with each Einstein-Maxwell spacetime $(M,\mathbf{g},\mathbf{F})$ an affine spacetime $(M,\{\ \}_g,-\mathbf{F})$ in which the Maxwell field tensor $\mathbf{F}$ plays the geometrical role of minus one times the R^4 connection $\hat{\mathbf{K}}$ in the zero-translation gauge. To complete the geometrization of the electromagnetic field we show in this section that the Maxwell equations (6.1) and (6.2) can be reformulated as geometrical equations in terms of the R^4 curvature. We postpone until the next section consideration of the coupling of the electromagnetic and gravitational fields through the Einstein equations.

Definition[10] *6.1.* An affine connection $(\{\ \}_g,{}^\theta\mathbf{K})$ is an electromagnetic affine connection if there exists an affine gauge $\hat{\beta}$ such that ${}^\beta\mathbf{K}$ is antisymmetric and satisfies the Maxwell equations (6.1) and (6.2).

A (generalized) affine connection $(\{\ \}_g,{}^\theta\mathbf{K})$ defines[3,11] a (generalized) affine curvature $(\mathbf{R},{}^\theta\mathbf{\Phi})$, where $\mathbf{R}$ is the Riemannian curvature tensor of $\{\ \}_g$, and ${}^\theta\mathbf{\Phi}$ is a rank-three tensor field representing the R^4 part of the curvature in the $\hat{\theta}$ translation gauge. The field ${}^\theta\mathbf{\Phi}$ can be expressed in terms of $(\{\ \}_g,{}^\theta\mathbf{K})$ by [cf. Eq. (4.10)]

$${}^\theta\Phi_{jk}{}^i=\nabla_j({}^\theta K^i{}_k)-\nabla_k({}^\theta K^i{}_j)\ . \tag{6.3}$$

Upon changing the origin field from $\hat{\theta}$ to $\hat{\Sigma}$ the fields ${}^\theta\mathbf{K}$ and ${}^\theta\mathbf{\Phi}$ transform into ${}^\Sigma\mathbf{K}$ and ${}^\Sigma\mathbf{\Phi}$, respectively, according to the rules

$${}^\Sigma K^j{}_k={}^\theta K^j{}_k+\nabla_k t^j\ , \tag{6.4}$$

$${}^\Sigma\Phi_{jk}{}^i={}^\theta\Phi_{jk}{}^i+R_{jkl}{}^i t^l\ , \tag{6.5}$$

where $\sigma\mathbf{t}=\delta(\hat{\Sigma},\hat{\theta})$. Note that these equations contain nonhomogeneous terms characteristic of the transformation laws for affine tensors.

Theorem 6.1. An affine connection $(\{\ \}_g,{}^\theta\mathbf{K})$ is an electromagnetic affine connection if and only if there exists a gauge $\hat{\beta}$ in which the following equations hold:

(a) ${}^{\beta}K_{(jk)}=0$,

(b) ${}^{\beta}\Phi_{[jki]}=0$,

(c) ${}^{\beta}\Phi_{jk}{}^{k}=0$.

Proof. If $(\{\ \}_g, {}^{\theta}\mathbf{K})$ is an electromagnetic affine connection then there exists a gauge $\hat{\beta}$ such that ${}^{\beta}\mathbf{K}=-\mathbf{F}$ where $\mathbf{F}$ is a Maxwell field tensor satisfying (6.1) and (6.2). Thus (a) is satisfied. Moreover, ${}^{\beta}\Phi_{jki} =-(\nabla_j F_{ki}-\nabla_k F_{ji})$ from which we get ${}^{\beta}\Phi_{[jki]} =-2\nabla_{[j}F_{ki]}=0$. Contracting ${}^{\beta}\Phi_{jk}{}^{i}$ we obtain ${}^{\beta}\Phi_{jk}{}^{k} =\nabla_k F_j{}^{k}=0$.

Conversely, suppose there exists a gauge $\hat{\beta}$ in which (a), (b), and (c) hold for an affine connection $(\{\ \}_g, {}^{\beta}\mathbf{K})$. Condition (a) requires ${}^{\beta}K_{jk}$ to be an antisymmetric tensor field. Then (b) and (c) together with the definition (6.3) show that ${}^{\beta}K_{jk}$ satisfies the Maxwell equations (6.1) and (6.2).

VII. GEOMETRIZATION OF THE EINSTEIN-MAXWELL EQUATIONS

We now come to the problem of geometrizing the full Einstein-Maxwell theory as a P(4) affine theory. We consider first an Einstein-Maxwell spacetime $(M,\mathbf{g},\mathbf{F})$ without sources. Then $\mathbf{g}$ and $\mathbf{F}$ are coupled together by the Einstein-Maxwell field equations

$$R_{jk}=8\pi k(F_{ji}F_k{}^{i}-\tfrac{1}{4}g_{jk}F_{mn}F^{mn}) , \quad (7.1)$$

$$\nabla_{[i}F_{jk]}=0 , \quad (7.2)$$

$$\nabla_j F^{jk}=0 . \quad (7.3)$$

In the Einstein-Maxwell theory only the metric tensor $\mathbf{g}$ from which the Ricci tensor R_{jk} is constructed is considered as a basic geometric quantity. The Maxwell field tensor $\mathbf{F}$ is regarded as an auxiliary tensor field whose energy-momentum tensor on the right-hand side of (7.1) serves as the source of the geometric gravitational field. Even in the "already unified theory" of Rainich, Misner, and Wheeler[12] the Maxwell field tensor is relegated to the role of the "Maxwell square root" of the Ricci tensor.

Using the results of the preceding sections it is now a simple matter to reformulate the source-free Einstein-Maxwell theory as a fully geometric P(4) affine theory. We replace $\mathbf{F}$ in (7.1) by $-({}^{\beta}\mathbf{K})$, and replace the Maxwell equations (7.2) and (7.3) with the geometric P(4) field equations given in theorem 6.1. Thus the field equations for an Einstein-Maxwell P(4) spacetime $(M,\{\ \}_g, {}^{\beta}\mathbf{K})$ in the absence of electric sources are

$$R_{jk}=8\pi k[{}^{\beta}K_{ji}({}^{\beta}K_k{}^{i})-\tfrac{1}{4}g_{jk}({}^{\beta}K_{mn})({}^{\beta}K^{mn})] , \quad (7.4)$$

$${}^{\beta}K_{(jk)}=0 , \quad (7.5)$$

$${}^{\beta}\Phi_{[ijk]}=0 , \quad (7.6)$$

$${}^{\beta}\Phi_{jk}{}^{k}=0 . \quad (7.7)$$

Furthermore, we postulate that the trajectories of classical test particles are the timelike and null geodesics of type σ, $\sigma\in R$, of the P(4) affine connection. When σ is nonzero the affine geodesic represents the trajectory of a particle with affine charge σ. Moreover, if m denotes the mass of the particle, then the electric charge q of a particle is defined by $q=m\sigma$ so that $m=0 \Longrightarrow q=0$. When $\sigma=0$ the affine geodesic represents the trajectory of an uncharged test particle.

When matter and current sources are present in spacetime then the above equations must be modified. A straightforward generalization is

$$G_{jk}=8\pi k[{}^{\beta}K_{ji}({}^{\beta}K_k{}^{i})-\tfrac{1}{4}g_{jk}({}^{\beta}K_{mn})({}^{\beta}K^{mn})+T_{jk}] , \quad (7.8)$$

$${}^{\beta}K_{(jk)}=0 , \quad (7.9)$$

$${}^{\beta}\Phi_{[ijk]}=0 , \quad (7.10)$$

$${}^{\beta}\Phi_{jk}{}^{k}=-J_j , \quad (7.11)$$

$$E(\{\theta^A\})=0 . \quad (7.12)$$

Equation (7.12) represents symbolically the equations for the matter source fields which we denote collectively by $\{\theta^A\}$. In Eq. (7.8) G_{jk} is the Einstein tensor and T_{jk} is the stress-energy tensor of the matter sources. The term J_j in (7.11) represents the electric source current of the matter.

Finally we show that the R^4-flat solutions of Eqs. (7.4)–(7.7) correspond to the covariant constant Maxwell fields that motivated the affine-vector model. If the R^4 curvature is integrable then ${}^{\beta}\Phi_{jk}{}^{i}=0$, and Eqs. (7.6) and (7.7) are satisfied. Then Eq. (6.3) together with (7.5) and (7.6) show that $\nabla_i({}^{\beta}K_{jk})=0$, so that the Maxwell field $F_{jk}=-({}^{\beta}K_{jk})$ is covariant constant.

VIII. DISCUSSION AND CONCLUSIONS

In this paper we have shown how to reinterpret the Riemannian geometry of an Einstein-Maxwell spacetime as a P(4) affine spacetime. The reinterpretation can be thought of as a completion of the geometrical unification of gravitation and electromagnetism that is only partially complete in the Einstein-Maxwell theory in which the geometry is Riemannian and the electromagnetic field plays only a secondary, nongeometrical role.

In the P(4) theory the Maxwell field tensor is identified with the R^4 part of the P(4) connection and is thus placed on a geometrical level with the Riemannian linear connection. The resulting P(4) unification of gravitation and electromagnetism led to a geometrization of the coupled Einstein-Maxwell field equations in terms of the O(1,3) and R^4 parts of the P(4) curvature. Moreover, in the P(4) theory the affine geodesics of type σ are the trajectories of test particles with charge-to-mass ratio σ that obey the Lorentz force law. The geometrical unification of gravitation and electromagnetism presented in this paper thus goes beyond the already unified field theory of Rainich, Misner, and Wheeler[12] in which the Maxwell field tensor is relegated to the role of the Maxwell square root of the Ricci tensor. Moreover, since the P(4) theory presented here is based on a P(4) connection it seems reasonable to expect that a variational principle can be found that would yield the geometrical Einstein-Maxwell affine field equations presented in Sec. VII. It should be recalled that a satisfactory variational principle for the Rainich-Misner-Wheeler theory has not yet been found.

The fundamental idea in the P(4) theory is to model the energy-momentum of a classical particle as an affine vector of type σ defined along the world line of the particle. If m denotes the mass of the particle, then the particle's electric charge q can be defined by $q = m\sigma$, thus guaranteeing the empirical law that $m = 0 \Longrightarrow q = 0$. The R^4 coupling parameter σ thus can be identified with q/m whenever $m \neq 0$.

On the physical side the energy-momentum translational freedom was related in Sec. III to the existence of an integrable electromagnetic potential energy-momentum in flat-Minkowski spacetime when the Maxwell field tensor is covariant constant. The covariant constant Maxwell fields were shown in Sec. VII to be precisely the R^4-flat solutions of the Einstein-Maxwell affine field equations (7.4)—(7.7). In addition, the usual notion of an instantaneous rest frame of a charged particle was related in Sec. IV to a local definition of the zero of energy-momentum.

The P(4) theory is in general agreement with the results of Boisseau and Barrabes.[13] Working within the Hamiltonian formalism they have shown that the only canonical transformations that can also be considered as gauge transformations lead to gauge fields that Boisseau and Barrabes identify with the gravitational and electromagnetic fields. The canonical transformations considered by Boisseau and Barrabes involve a general coordinate transformation together with a translation of the canonical energy-momentum; however, Boisseau and Barrabes do not relate their work to P(4) affine geometry.

A number of questions about the P(4) theory presented in this paper remain open. In the P(4) theory the Maxwell field tensor enters the theory geometrically as the R^4-connection coefficients. In this sense the P(4) theory is parallel with the five-dimensional Kaluza-Klein theory[14] in which the Maxwell field tensor appears as certain of the components of the five-dimensional Riemannian linear connection. Does there exist a fundamental relationship between the P(4) theory presented here and the theory of Kaluza and Klein? A more important question concerns the gauge invariance of the Einstein-Maxwell affine field equations (7.4)—(7.7). In Secs. V—VII we have fixed the gauge by introducing the zero-translation gauge $\hat{\boldsymbol{\beta}}$, and Eqs. (7.4)—(7.7) are given in terms of this gauge. The transformation laws (6.4) and (6.5) show that the form of Eqs. (7.4), (7.5), and (7.7) are not gauge invariant. It is remarkable, however, that Eq. (7.6) (the $\mathbf{F}=d\mathbf{A}$ equation) is gauge invariant. This follows from (6.5) and the Riemannian curvature identity $R_{(ijk)}{}^{l}=0$. Can Eqs. (7.4)—(7.7) be generalized to a translationally covariant set of equations without the need for the gauge-fixing condition? The answer to this question will be useful in searching for a variational principle for the theory. We hope to return to these and other related questions in future publications.

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[1]Throughout this paper we will follow the convention that tensorial quantities will be denoted by boldface letters, while affine quantities will also be denoted by boldface letters but with carets (e.g., $\hat{\mathbf{u}}$). The standard latin superscripts and subscripts written to the right of the letter will be used to denote the linear components of tensorial quantities. When an affine quantity, say, $\hat{\mathbf{u}}$, is referred to coordinates it will need an additional superscript to indicate the origin field to which it is referred, and this superscript will be written to the left of the letter. Thus ${}^{\beta}u^{j}$ will denote the components of $\hat{\mathbf{u}}$ when referred to an origin field $\hat{\boldsymbol{\beta}}$ and a linear frame $(\mathbf{e}_j)$.

[2]For an introduction to affine spaces see, for example, C. T. J. Dodson and T. Poston, *Tensor Geometry* (Pitman, London, 1977).

[3]For a general discussion of the differential geometry of generalized affine connections, see S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963), Vol. I; A. Lichnerowicz, *Global Theory of Connections and Holonomy Groups*, edited by M. Cole (Noordhoff International, Leyden, 1976).

[4]The absolute derivative of an affine vector is a vector. This follows intuitively from noting that the very definition of the derivative involves the limit of the difference of two affine quantities, and the difference of two affine vectors is a tensorial quantity.

[5]Definitions 2.1 and 3.1 are modeled after similar definitions in R. K. Sachs and H. Wu, *General Relativity for Mathematicians* (Springer, New York, 1977).

[6]The affine connection for an R^3 gauge theory is a three-vector-valued one-form $\mathbf{B}=B^{j}{}_{k}\partial_j \otimes dx^k$. Given a Newtonian force $\mathbf{v}\times\mathbf{B}$ one may define $B^{j}{}_{k}=-B^{i}\epsilon_{ik}{}^{j}$. Then the affine covariant derivative of an affine three-vector $\hat{p}^{\,j}$ along a curve $x^{j}(t)$ is

$$D\hat{p}^{\,j}/Dt = dp^{j}/dt + B^{j}{}_{k}(dx^{k}/dt)$$
$$= dp^{j}/dt - B^{i}\epsilon_{ik}{}^{j}(dx^{k}/dt)\ .$$

Then $D\hat{\mathbf{p}}/Dt=0 \Longrightarrow d\mathbf{p}/dt=\mathbf{v}\times\mathbf{B}$.

[7]The vector potential A_j for a covariant constant Maxwell field $F_{jk}=\partial_j A_k-\partial_k A_j$ in Minkowski spacetime can be chosen as $A_j=\frac{1}{2}F_{kj}x^{k}$. The potential energy-momentum $q\theta^{j}$ may thus be expressed as $q\theta^{j}=2qA^{j}$ in this special case. Such a simple relationship between the R^4 potential and the vector potential will not hold if F_{jk} is not covariant constant (cf. Sec. V).

[8]The instantaneous rest frame used here is roughly equivalent to the "instantaneous observer" in R. K. Sachs and H. Wu, *General Relativity for Mathematicians* (Ref. 5) and to the "comoving inertial frame" in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[9]For arbitrary $\sigma\neq 0$ the σ-affine-vector fields discussed in Sec. V generalize the "point fields" ($\sigma=1$) in S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Ref. 3).

[10]This definition of "electromagnetic affine connection" requires the linear part of the affine connection to be Riemannian. If one contemplated a theory in which the linear geometry was more general than Riemannian, then this definition would have to be modified.

[11]For a general discussion of the structure of A(4) gauge theories see L. K. Norris, R. O. Fulp, and W. R. Davis, Phys. Lett. **79A**, 278 (1980).

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