GENERALIZED SYMPLECTIC GEOMETRY

ON THE

FRAME BUNDLE OF A MANIFOLD†

by

L. K. Norris

Department of Mathematics
Box 8205
North Carolina State University
Raleigh, NC 27695-8205

†Proceedings of Symposia in Pure Mathematics, **54**, Part 2, pp. 435–465 (1993)

ABSTRACT

In this paper we develope the fundamentals of the generalized symplectic geometry on the bundle of linear frames LM of an n-dimensional manifold M that follows upon taking the $\mathbb{R}^n$-valued soldering 1-form θ on LM as a generalized symplectic potential. The development is centered around generalizations of the basic structure equation $df = -X_f \rfloor \omega$ of standard symplectic geometry to LM when the symplectic 2-form ω is replaced by the closed and non-degenerate $\mathbb{R}^n$-valued 2-form $\beta = d\theta = d\theta^i r_i$. The fact that $d\theta$ is $\mathbb{R}^n$-valued necessitates generalizing from $\mathbb{R}$-valued observables to vector-valued observables on LM, and there is a corresponding increase in the number of Hamiltonian vector fields assigned to each observable. We show that the algebras of symmetric and anti-symmetric contravariant tensor fields on the base manifold have natural interpretations in terms of symplectic geometry on LM. For the analysis we consider in place of each rank p contravariant tensor field on the base manifold the uniquely related $\otimes^p\mathbb{R}^n$-valued tensorial function on LM. For symmetric contravariant tensor fields on M we show that the associated algebra $(ST, \otimes_s)$, where $ST = \sum_{p=1}^{\infty} ST^p$ is the vector space of all $\otimes_s^p\mathbb{R}^n$-valued tensorial functions on LM, becomes a Poisson algebra under a generalized Poisson bracket. In addition the associated set of locally defined $\otimes_s^{p-1}\mathbb{R}^n$-valued Hamiltonian vector fields $\hat{X}_{\hat{f}}$ forms a Lie algebra under a generalized Lie bracket. In the case of anti-symmetric contravariant tensor fields on M we show that the corresponding vector space $AT = \sum_{p=1}^{\infty} AT^p$ of $\otimes_a^p\mathbb{R}^n$-valued functions on LM becomes a Poisson super algebra under a naturally defined bracket. The associated set of locally defined $\otimes_a^{p-1}\mathbb{R}^n$-valued Hamiltonian vector fields $\hat{X}_{\hat{f}}$ forms a super algebra under a generalized super bracket. The naturally defined brackets of the tensorial functions on LM give the Schouten differential concomitants when reinterpreted on the base manifold. Generalized symplectic geometry on the frame bundle of a manifold thus unifies and clarifies the many different approaches to the differential concomitants of Schouten. Two applications of the geometry to physics are presented. First the dynamics of free inertial observers in spacetime is shown to follow upon taking the metric tensor as the Hamiltonian for free observers. We then show that the Dirac equation arises in a natural way as an eigenvalue equation for a naive prequantization operator assigned to the spacetime metric tensor Hamiltonian.

1. Introduction

The methods introduced by W.R. Hamilton more than a century and a half ago have since been used fundamentally in physics in the development of classical and quantum mechanics and the classical and quantum theory of fields. During the last three decades Hamilton's methods have been given a beautiful, invariant and geometrical formulation in the theory of symplectic geometry, the fundamentals of which can be found in the works of Sternberg [1], Hermann [2], Arnol'd [3], Abraham and Marsden [4], and Guillemin and Sternberg [5].

Since its inception symplectic geometry has served to motivate a number of new developments. In particular the formulation of Hamiltonian dynamics in terms of symplectic geometry is the starting point for the theory of geometric quantization due to Kostant [6] and Souriau [7]. Generalizations of symplectic geometry have led to the study of canonical manifolds [8], Poisson manifolds [9,10,11], and to the idea of Poisson algebras [11,12]. A recent account of the applications of symplectic techniques in a wide variety of physical problems, including Yang–Mills theory, can be found in the book by Guillemin and Sternberg [13].

This paper introduces, and develops aspects of, a generalized symplectic geometry on the bundle of linear frames of a manifold. Mathematical motivation for this generalization may be provided in the following way. The canonical model of a symplectic manifold is the pair $(T^*M, d\tilde{\theta})$, where T^*M is the cotangent bundle of an n-dimensional differentiable manifold M, and $\tilde{\theta}$ is the **canonical 1-form** on T^*M that plays the role of a globally defined symplectic potential. The point to be emphasized here is that one obtains $\tilde{\theta}$ "for free" from the differential structure of T^*M.

Now consider T^*M as the fiber bundle $LM \times_{GL(n)} (\mathbb{R}^n)^*$ associated to the bundle of linear frames LM of M and the standard action of $GL(n) \equiv GL(n,\mathbb{R})$ on $(\mathbb{R}^n)^*$. From this point of view it is clear that T^*M inherits its differential structure from LM. This being the case one is led to ask if the canonical symplectic structure on T^*M has its roots in a more general structure on LM. The natural candidate for a symplectic potential on LM is the $\mathbb{R}^n$-valued **soldering 1-form** θ on LM which also comes "for free" from the differential structure of LM. Here we note that the exact 2-form $d\theta$ is non-degenerate in the sense that

$$X \lrcorner \; d\theta = 0 \;\Leftrightarrow X = 0$$

for X a vector field on LM. Moreover, if dim(M)=n then the dimension of the frame bundle LM is the even number n(n+1). Thus we would have the necessary ingredients for a symplectic manifold if it were not for the fact that $d\theta$ is $\mathbb{R}^n$-valued. Nonetheless we make the following

DEFINITION: The $\mathbb{R}^n$-valued two form $d\theta$ on LM is a **generalized symplectic structure**. The pair $(LM, d\theta)$ will be referred to as a **generalized symplectic manifold**.

The geometry that one can build up from this definition is more general and at the same time more special than standard symplectic geometry. It is more general in the sense that the $\mathbb{R}$-valued observables on T^*M are replaced by **vector-valued** functions on LM, together with an increase in the number of associated Hamiltonian vector fields. On the other hand the geometry is more special in the sense that while in principle an arbitrary $\mathbb{R}$-valued function on T^*M is an allowable observable, not all vector-valued functions on LM are compatible with the geometry. Both of these features will be seen to be due to the fact that the general linear group $GL(n)$ acts on the manifold LM, and that the soldering 1-form θ transforms **tensorially** under this action. Consequently the set of all allowable observables contains the vector-valued tensorial functions on LM that correspond uniquely to tensor fields on M. Although more general observables may be compatible with the geometry, in this paper we will restrict attention to the geometry associated with vector-valued observables related to the standard actions of $GL(n)$ on $\mathbb{R}^n$ and $\mathbb{R}^{n*}$.

Physical motivation for this generalized geometry comes from the observation that the bundle of linear frames LM is the bundle that is fundamentally related to spacetime observations. This fact coupled with the inescapable interaction of observer and object that is a basic feature of quantum mechanics supports the idea that symplectic geometry on LM may be useful in quantum theory. In fact we show in Section 8 that the Dirac equation arises in a natural way as the eigenvalue equation for a generalized pre-quantization operator assigned to the spacetime metric tensor Hamiltonian on LM. This new result lends strong support to this study of generalized symplectic geometry.

In the first few sections of this paper we develop the fundamentals of this symplectic geometry on LM. After providing a motivational algorithm for the fundamental structure equation in Section 2, we show in Section 3 that symmetric contravariant tensor fields on the base manifold M give rise to a **Poisson algebra** on LM. We show that the Poisson bracket of two tensorial functions $\hat{f}$ and $\hat{g}$ on LM, corresponding to tensor fields $\vec{f}$ and $\vec{g}$ on M, is the tensorial function corresponding to the differential concomitant of $\vec{f}$ and $\vec{g}$ discovered by Schouten and Nijenhuis [14,15]. The corresponding sets of locally defined vector-valued Hamiltonian vector fields define Lie algebras on LM.

Extending the analysis to anti-symmetric contravariant tensor fields on M we show in Section 4 that such fields on M give rise to a **Poisson super-algebra** on LM. The super algebra bracket again reproduces the differential concomitants of Schouten and Nijenhuis [14,15] for two anti-symmetric contravariant tensor fields. The corresponding sets of locally defined vector-valued Hamiltonian vector fields define super algebras on LM.

In Section 5 we use a result from Section 2, that the **natural lift** of a vector field on M to LM is a rank $p = 1$ Hamiltonian vector field on LM, to give an extension of the definition to contravariant rank > 1 tensor fields. We also show in Section 5 how the Poisson bracket introduced in Sections 2 and 3 leads to a natural definition of Killing tensors associated with a given Riemannian metric tensor field. In Section 6 we define **locally Hamiltonian vector fields** in the context of generalized symplectic geometry and study the associated integrability conditions. We show that the most general $\otimes_s^p \mathbb{R}^n$-valued functions on LM that are compatible with the geometry are generalized **polynomial observables**, that is $\otimes_s^p \mathbb{R}^n$-valued polynomials in the generalized momentum coordinates π^i_j with coefficients in the set of $\mathbb{R}$-valued functions on the base space M. The analogous result in the anti-symmetric case is obtained by replacing "polynomial in π^i_j" with "exterior products of the π^i_j".

In Sections 7 and 8 we provide two applications of the geometry to physics. The equations of motion of free inertial observers are derived in Section 7, while in Section 8 we derive the Dirac equation from a generalized geometric pre-quantization argument. Finally in Section 9 we present a summary and conclusions.

It is convenient to introduce here a portion of the notation that will be needed in the remainder of the paper. Let U be an open subset of M with n=dim(M). Set $\hat{U} = \pi^{-1}(U)$ where $\pi : LM \to M$ is the projection map. All actions of $GL(n) \equiv GL(n,\mathbb{R})$ on the spaces $\otimes^p \mathbb{R}^n$, $p = 1, 2, \ldots$, indicated below by a central dot " $\cdot$ ", are the standard tensorial actions. We use the abbreviated notation $\otimes_s^p \mathbb{R}^n$ for the p-fold symmetric tensor product $\mathbb{R}^n \otimes_s \mathbb{R}^n \otimes_s \cdots \otimes_s \mathbb{R}^n$, and $\otimes_a^p \mathbb{R}^n$ for the p-fold anti-symmetric tensor product $\mathbb{R}^n \otimes_a \mathbb{R}^n \otimes_a \cdots \otimes_a \mathbb{R}^n$.

- $ST^p = \{\hat{f} : LM \to \otimes_s^p \mathbb{R}^n \mid \hat{f}(u \cdot g) = g^{-1} \cdot \hat{f}(u) \ \forall g \in GL(n)\}$ is the vector space of symmetric $\otimes_s^p \mathbb{R}^n$-valued tensorial functions on LM. An element of ST^p corresponds to a unique rank p symmetric contravariant tensor field on M.

- $AT^p = \{\hat{f} : LM \to \otimes_a^p \mathbb{R}^n \mid \hat{f}(u \cdot g) = g^{-1} \cdot \hat{f}(u) \ \forall g \in GL(n)\}$ is the vector space of anti-symmetric $\otimes_a^p \mathbb{R}^n$-valued tensorial functions on LM. An element of AT^p corresponds to a unique rank p anti-symmetric contravariant tensor field on M.

- $ST = \sum_{p=1}^{\infty} ST^p$
- $AT = \sum_{p=1}^{\infty} AT^p$
- $\mathcal{X}(N)$ denotes the vector space of smooth vector fields on a differentiable manifold N.
- $S\mathcal{X}^p \equiv S\mathcal{X}^p(N)$ denotes the vector space of smooth symmetric contravariant tensor fields on N of rank p.
- $A\mathcal{X}^p \equiv A\mathcal{X}^p(N)$ denotes the vector space of smooth anti-symmetric contravariant tensor fields on N of rank p.
- $S\mathcal{X} = \sum_{p=1}^{\infty} S\mathcal{X}^p$
- $A\mathcal{X} = \sum_{p=1}^{\infty} A\mathcal{X}^p$

2. Generalized Symplectic Geometry

Let M be an n-dimensional manifold and LM the principal fiber bundle of linear frames of M. The dimension of LM is the even number $n(n+1)$. A point $u \in LM$ will be denoted by the pair (p, e_i) where $p \in M$ and $(e_i) \equiv (e_1, e_2, \ldots, e_n)$ denotes a linear frame at p. The projection map $\pi : LM \to M$ is defined by $\pi(p, e_i) = p$. The structure group of LM is the general linear group $GL(n)$, which acts freely on the right of LM by $R_g(p, e_i) \equiv (p, e_i) \cdot g = (p, e_j g^j_i)$ for $g = (g^i_j) \in GL(n)$. In this definition and throughout this paper the summation convention on repeated indices is employed.

Local coordinates on LM may be defined as follows. If (U, x^i) is a chart on M, then define local coordinates $(x^i, \pi^j_k) : \pi^{-1}(U) \to \mathbb{R}^n \times \mathbb{R}^{n^2}$ by

$$\begin{aligned} x^i(p, e_j) &= x^i(p) \ , \\ \pi^j_k(p, e_i) &= e^j\left(\frac{\partial}{\partial x^k}\right) \ . \end{aligned} \tag{2.1}$$

In this definition (e^j) , $j = 1, 2, \ldots, n$ denotes the coframe dual to the linear frame (e_j). Moreover we follow the standard practice of using x^i to denote coordinates on both $U \subset M$ and $\pi^{-1}(U) \subset LM$.

The structure of LM is special in the sense that it supports a globally defined $\mathbb{R}^n$-valued 1-form, the **soldering 1-form** $\theta = \theta^i r_i$. Here $r_1, r_2, \ldots, r_n$ denotes the standard basis of $\mathbb{R}^n$. For each point $u \in LM$ let u also denote the linear map $u : \mathbb{R}^n \rightarrow T_{\pi(u)}M$ defined by [16]

$$u(\xi^i r_i) \equiv (p, e_j)(\xi^i r_i) \stackrel{def}{=} \xi^i e_i \quad , \tag{2.2}$$

with inverse

$$u^{-1}(X) \equiv (p, e_i)^{-1}(X) = e^i(X) r_i \quad , \quad X \in T_p M \ . \tag{2.3}$$

Then the soldering 1-form θ may be defined by

$$\theta(Y) \stackrel{def}{=} u^{-1}(d\pi Y) \quad , \quad \forall \, Y \in T_u LM \ . \tag{2.4}$$

In local coordinates (x^i, π^j_k) the soldering 1 form has the local expression

$$\theta^i r_i = (\pi^i_j dx^j) r_i \ . \tag{2.5}$$

One may compare this form to the expression $\tilde{\theta} = \pi_j dx^j$ for the canonical 1-form on T^*M in canonical coordinates.

The basic properties of θ that follow from its definition are

$$\begin{aligned} &(a) \quad \theta(Y) = 0 \Leftrightarrow d\pi(Y) = 0 \ , \\ &(b) \quad R^*_g \theta = g^{-1} \cdot \theta \equiv (g^{-1})^i_j \theta^j r_i \ . \end{aligned} \tag{2.6}$$

The 1-form θ is the basic element needed to define the torsion Θ of a linear connection. If ω denotes the $gl(n)$-valued 1-form of a linear connection on LM, then the torsion of ω may be defined by [16]

$$\begin{aligned} \Theta &= d\theta + \omega \wedge \theta \\ &= (d\theta^i + \omega^i_j \wedge \theta^j) r_i \ . \end{aligned} \tag{2.7}$$

In particular, if a linear connection ω is torsion-free then

$$d\theta^i = -\omega^i_j \wedge \theta^j \ . \tag{2.8}$$

By the Frobenius theorem the n-dimensional co-distribution spanned globally by the 1-forms θ^i is integrable, and the integral submanifolds of the codistribution are clearly the fibers of LM.

Consider now the exact $\mathbb{R}^n$-valued 2-form $\beta \stackrel{def}{=} d\theta$. By (2.5) it has the local coordinate expression

$$\beta = \beta^i r_i = (d\pi^i_j \wedge dx^j) r_i \ . \tag{2.9}$$

Using this last equation (or equation (2.8)) it is easy to show that β is **non-degenerate** in the sense that

$$X \rfloor \beta = 0 \Leftrightarrow X = 0 \ . \tag{2.10}$$

In standard symplectic geometry on T^*M one uses the canonical 1-form $\tilde{\theta}$ to assign a unique Hamiltonian vector field X_f to each observable $f : T^*M \to \mathbb{R}$ via the equation

$$df = -X_f \rfloor \ d\tilde{\theta} \quad . \tag{2.11}$$

If we attempt to transcribe this equation to LM using the soldering 1-form we have

$$df = -X_f \rfloor \ d\theta^i r_i \quad ,$$

and it is clear that this expression makes no sense for f a $\mathbb{R}$-valued function and X_f a vector field on LM. However, if we replace f with an $\mathbb{R}^n$-valued function $\hat{f} = \hat{f}^i r_i : LM \to \mathbb{R}^n$, then the equation

$$d\hat{f} = -X_{\hat{f}} \rfloor \ d\theta^i r_i \tag{2.12}$$

defines a unique vector field $X_{\hat{f}}$ given $\hat{f}$. In order to facilitate the derivation of other generalizations of equation (2.11) to LM it is convenient to introduce the following geometrical derivation of equation (2.12).

Consider the problem of finding a torsion–free linear connection ω on LM with respect to which a given vector field $\vec{f}$ on the base manifold M is covariant constant. Let $\hat{f} = \hat{f}^i r_i$ denote the unique $\mathbb{R}^n$-valued tensorial 0-form on LM determined by $\vec{f}$, defined invariantly by $\hat{f}(u) = u^{-1}(\vec{f}(\pi(u)))$. The covariant derivative of $\vec{f}$ on M is uniquely determined [16] by the exterior covariant derivative $D\hat{f} = (d\hat{f}^i + \omega^i_j \cdot \hat{f}^j) r_i$ of $\hat{f}$ on LM . Let (B_i) , $i = 1, 2, \ldots, n$ denote the standard horizontal vector fields on LM determined by ω. Then these vector fields satisfy

$$\begin{aligned} \omega(B_i) &= 0 \ , \\ \theta^i(B_j) &= \delta^i_j \ , \end{aligned} \tag{2.13}$$

for $i, j = 1, 2, \ldots, n$.

From equations (2.8) and (2.13) we find for a torsion-free linear connection the relationship

$$\omega^i_j = B_j \rfloor \ \beta^i \ . \tag{2.14}$$

If this expression for ω^i_j is substituted into the formula for the exterior covariant derivative of $\hat{f}$ then the $\mathbb{R}^n$ components of $D\hat{f}$ may be expressed as

$$D\hat{f}^i = d\hat{f}^i + (B_j \hat{f}^j) \rfloor \ \beta^i \quad . \tag{2.15}$$

Defining $X_{\hat{f}} \stackrel{def}{=} B_j \hat{f}^j$ and assuming $D\hat{f} = 0$ we obtain the equation

$$d\hat{f}^i = -X_{\hat{f}} \rfloor \beta^i \ . \tag{2.16}$$

Given the functions $\hat{f}^i$ the vector field $X_{\hat{f}}$ is uniquely determined by equation (2.16) since $\beta = \beta^i r_i$ is non degenerate. Thus solutions to the original problem can be sought by first solving equation (2.16) for $X_{\hat{f}}$ and by then solving $X_{\hat{f}} = B_i \hat{f}^i$ for the vector fields (B_i), which would certainly not be unique. If the vector fields (B_i) can be made to satisfy certain additional conditions then they would define a linear connection with the required property.

The purpose of this example in not to discuss the existence or uniqueness of such a linear connection, but rather to bring to light equation (2.16). If we now disregard the method of derivation of equation (2.16) and the definition of $X_{\hat{f}}$, then it is clear that $\beta = \beta^i r_i$ plays the role of a generalized symplectic structure on LM. We may think of $X_{\hat{f}}$ as the *generalized Hamiltonian vector field* determined by the $\mathbb{R}^n$-valued tensorial function $\hat{f}$, and we may consider the flow of $X_{\hat{f}}$ as generating local one-parameter families of *generalized canonical transformations.* These transformations are canonical in the sense that $\mathcal{L}_{X_{\hat{f}}}(\beta) = 0$, which follows in the standard way from equation (2.16) and the general formula $\mathcal{L}_X \Psi = X \rfloor d\Psi + d(X \rfloor \Psi)$.

When the functions $\hat{f}^i$ are determined as above from a vector field $\vec{f}$ on M, then the Hamiltonian vector field $X_{\hat{f}}$ determined from equation (2.16) will be shown below to be the **natural lift** of $\vec{f}$ to LM. This point is important because, since the natural lift is independent of any connection on LM, it shows the basic independence of equation (2.16) from ideas of covariant differentiation based on linear connections. Explicitly, let $\vec{f}$ be given in local coordinates on M by

$$\vec{f} = f^i \frac{\partial}{\partial x^i} \tag{2.17}$$

so that the corresponding function $\hat{f}$ on LM is given by

$$\hat{f} = \hat{f}^i r_i = ((f^j \circ \pi)\pi^i_j) r_i \ . \tag{2.18}$$

Note that under right translation on LM the functions $\hat{f}^i$ transform according to the rule $\hat{f}^i(u \cdot g) = (g^{-1})^i_j \hat{f}^j(u)$ for $u \in LM$ and $g = (g^i_j) \in GL(n)$. This is the *tensorial* transformation law [16] for $\mathbb{R}^n$-valued functions on LM.

Solving equation (2.16) locally for $X_{\hat{f}}$ with $\hat{f}$ as in (2.18) yields

$$X_{\hat{f}} = (f^i \circ \pi)\frac{\partial}{\partial x^i} - (\frac{\partial (f^i \circ \pi)}{\partial x^j}\pi^k_i)\frac{\partial}{\partial \pi^k_j} \ . \tag{2.19}$$

Using this result it is easy to show that $X_{\hat{f}}$ has the following three properties:

$$\begin{aligned}
&(1) \quad dR_a(X_{\hat{f}}) = X_{\hat{f}} \quad \text{for every } a \in GL(n) \ , \\
&(2) \quad \mathcal{L}_{X_{\hat{f}}}(\theta) = 0 \ , \\
&(3) \quad d\pi(X_{\hat{f}}) = \vec{f} \ .
\end{aligned} \tag{2.20}$$

Properties (1) and (3) follow from (2.19), while property (2) follows from equation (2.16). These three properties uniquely characterize [16] the **natural lift** of a vector field on M to LM. It follows that the canonical transformations generated by the flow of $X_{\hat{f}}$ on LM represent the natural lift to LM of the local diffeomorphisms of M generated by the flow of $\vec{f}$. We have the result that the natural dynamics of vector fields on a manifold M is *Hamiltonian dynamics with respect to the symplectic structure $d\theta$ on LM*. We formalize these results in the following

Theorem 2.1: Let $\hat{f} : LM \to \mathbb{R}^n$ be the tensorial 0-form on LM determined by a vector field $\vec{f}$ on M. Then the Hamiltonian vector field $X_{\hat{f}}$ determined by

$$d\hat{f} = -X_{\hat{f}} \rfloor \ \beta$$

is the natural lift of $\vec{f}$ to LM.

Modifications of standard symplectic geometry begin to appear when equation (2.16) is examined more closely. The first thing to notice is that while there are no restrictions placed on the $\mathbb{R}$-valued functions f on T^*M by equation (2.11), not every $\mathbb{R}^n$-valued function on LM is compatible with equation (2.16). Let:

- $\mathrm{HF}^1 \equiv \mathrm{HF}^1(LM,\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$-valued functions on LM that satisfy equation (2.16) for some vector field $X_{\hat{f}}$,
- HV^1 denote the set of Hamiltonian vector fields determined by HF^1 and equation (2.16),
- $\mathrm{T}^1 \equiv \mathrm{T}^1(LM,\mathbb{R}^n)$ denote the set of $\mathbb{R}^n$-valued tensorial 0-forms on LM relative to the standard action of GL(n) on $\mathbb{R}^n$.
- $\mathrm{LHF}^1 \equiv \mathrm{HF}^1(\pi^{-1}(U),\mathbb{R}^n)$, U an open subset of M, denotes the set of locally defined $\mathbb{R}^n$-valued functions on LM that satisfy equation (2.16) for some vector field $X_{\hat{f}}$ on $\pi^{-1}(U)$.

An analysis of equation (2.16) (see Section 6) shows that the locally defined set of $\mathbb{R}^n$-valued functions LHF^1 consists of functions of the form

$$\begin{aligned} \hat{f} &= \hat{f}^i r_i \\ &= \{(f^i \circ \pi)\pi_i^j + \xi^j \circ \pi\} r_j \ , \end{aligned} \tag{2.21}$$

where f^i and ξ^i are functions defined on $U \subset M$. Thus, upon comparing (2.21) with (2.18) we have

$$\mathrm{LHF}^1 = \mathrm{T}^1(\pi^{-1}(U), \mathbb{R}^n) + C^\infty(U, \mathbb{R}^n) \quad . \tag{2.22}$$

The Hamiltonian vector field $X_{\hat{f}}$ determined locally by such an element of LHF^1 has the local expression

$$X_{\hat{f}} = (f^i \circ \pi)\frac{\partial}{\partial x^i} - (\frac{\partial (f^i \circ \pi)}{\partial x^k}\pi_i^j + \frac{\partial(\xi^j \circ \pi)}{\partial x^k})\frac{\partial}{\partial \pi_k^j} \ . \tag{2.23}$$

It is straight forward to show that $\mathbf{HF}^1$ **is a Lie algebra** under the bracket defined by

$$\{\hat{f}, \hat{g}\} \stackrel{def}{=} X_{\hat{f}}(\hat{g}) \quad . \tag{2.24}$$

Moreover, by direct calculation one may show that $\mathbf{HV}^1$ **is a Lie algebra** under Lie bracket, and that $[X_{\hat{f}}, X_{\hat{g}}] = X_{\{\hat{f},\hat{g}\}}$.

The explicit local expression for the Poisson bracket of $\hat{f} = \{(f^i \circ \pi)\pi_i^j + \xi^j \circ \pi\} r_j$ and $\hat{g} = \{(g^i \circ \pi)\pi_i^j + \eta^j \circ \pi\} r_j$ is

$$\{\hat{f}, \hat{g}\} = \{f^i \frac{\partial g^k}{\partial x^i} - g^i \frac{\partial f^k}{\partial x^i}\}\pi_k^j r_j + \{f^i \frac{\partial \eta^j}{\partial x^i} - g^i \frac{\partial \xi^j}{\partial x^i}\} r_j \ . \tag{2.25}$$

Therefore the sum in (2.22) is a semi-direct sum.

The center of LHF^1 consists of the constant functions $\pi^{-1}(U) \to \mathbb{R}^n$, so that as Lie algebras

$$\mathrm{LHV}^1 \simeq \mathrm{LHF}^1/\mathbb{R}^n \quad .$$

LHF^1 can thus be regarded as a **central extension** of LHV^1, in general agreement with the standard theory on T^*M.

3. The Poisson Algebra ST on LM

As shown above the Lie bracket of vector fields on M is equivalent to the Lie bracket of tensorial $\mathbb{R}^n$-valued functions on LM defined in equation (2.24). Now the Lie bracket of vector fields is a derivation on the space of vector fields, and it is well-known to have an extension to derivations of arbitrary tensor fields. However, the Lie derivative $\mathcal{L}_{\vec{f}}(T)$ of a rank $p > 1$ contravariant tensor field T with respect to a vector field $\vec{f}$ is not a Lie bracket, and hence $\mathcal{L}_{\vec{f}}(T)$ would not seem to have any relationship to a Poisson bracket. However we will show that there is a relationship for certain classes of irreducible tensors.

A symmetric rank p contravariant tensor field $\vec{g} \in \mathrm{S}\mathcal{X}^p$ has the local coordinate expression

$$\vec{g} = g^{i_1 \dots i_p} \partial_{i_i} \otimes \cdots \otimes \partial_{i_p} \quad , \quad g^{i_1 \dots i_p} = g^{(i_1 \dots i_p)} \ . \tag{3.1}$$

In this equation and in the following "round" brackets on indices denotes symmetrization. The corresponding function $\hat{g} \in \mathrm{ST}^p$ is given in local coordinates (x^i, π^j_k) by

$$\begin{aligned} \hat{g} &= \hat{g}^{i_1 \dots i_p} r_{i_1} \cdots r_{i_p} \\ &= (g^{j_1 \dots j_p} \circ \pi) \pi^{i_1}_{j_1} \cdots \pi^{i_p}_{j_p} r_{i_1} \otimes \cdots \otimes r_{i_p} \ . \end{aligned} \tag{3.2}$$

Now consider $\hat{f} \in \mathrm{ST}^1$ and $\hat{g} \in \mathrm{ST}^2$. Let $X_{\hat{f}}$ denote the Hamiltonian vector field associated with $\hat{f}$. By analogy with equation (2.24) one is led to try the definition

$$\{\hat{f}, \hat{g}\} \stackrel{def}{=} X_{\hat{f}}(\hat{g}) = X_{\hat{f}}(\hat{g}^{ij}) r_i \otimes r_j \tag{3.3}$$

for the Poisson bracket of $\hat{f}$ with $\hat{g}$. It is straight forward to check using (2.19) that the term $X_{\hat{f}}(\hat{g})$ on the right hand side of this definition is in fact the element of ST^2 corresponding to the Lie derivative of the associated tensor field $\vec{g}$ on M with respect to the vector field $\vec{f}$. This is the LM form of the extension of the Lie derivative mentioned above. The problem of course with the definition (3.3) is how to make sense of

$$\{\hat{g}, \hat{f}\} = -\{\hat{f}, \hat{g}\} \tag{3.4}$$

so that we actually have a Poisson bracket.

So far we have only associated Hamiltonian vector fields with elements of ST^1. To give meaning to the left hand side of equation (3.4) we need to associate Hamiltonian vector fields with all elements of ST. A method for doing this can be found by considering again the derivation of equation (2.16), which we illustrate for an element $\hat{g} \in \mathrm{ST}^2$. The derivation starts with the problem of finding a torsion-free linear connection that leaves $\hat{g}$ covariant constant. The generalization of equation (2.16) for this problem is

$$\begin{aligned} d\hat{g} &= (d\hat{g}^{ij}) r_i \otimes r_j \\ &= (-X^j_{\hat{g}} \rfloor \beta^i - X^i_{\hat{g}} \rfloor \beta^j) r_i \otimes r_j \ , \end{aligned}$$

or simply

$$d\hat{g}^{ij} = -2X^{(i}_{\hat{g}} \rfloor \beta^{j)} \ . \tag{3.5}$$

Thus a symmetric $\mathbb{R}^n \otimes \mathbb{R}^n$-valued tensorial function on LM has associated with it a **set of Hamiltonian vector fields** $X^i_{\hat{g}}$, $i = 1, 2, \ldots, n$ rather than a single Hamiltonian vector field. Taken together these vector fields $X^i_{\hat{g}}$ define the $\mathbb{R}^n$-valued vector field $\hat{X}_{\hat{g}} = X^i_{\hat{g}} \otimes r_i$.

Now although β is nondegenerate in the sense of equation (2.10), because of the symmetrization in equation (3.5) $\hat{X}_g$ is not uniquely determined by $\hat{g}$ and equation (3.5). However, the non-uniqueness is easily characterized, at least locally. In particular, an element $\hat{g} \in \mathrm{ST}^2$ determines n vector fields X^i_g via equation (3.5) up to addition of vector fields Y^i on LM satisfying the kernel equation

$$Y^{(i} \rfloor \beta^{j)} = 0 \ .$$

More generally, an element $\hat{g} \in \mathrm{ST}^p$ as in (3.2) above determines $N_S(p) = \begin{pmatrix} n+p-2 \\ p-1 \end{pmatrix}$ vector fields $X^{i_1 \ldots i_{p-1}}_g$ via the generalized symplectic structure equation

$$d\hat{g}^{i_1 \ldots i_p} = -p! X^{(i_1 \ldots i_{p-1}}_{\hat{g}} \rfloor \beta^{i_p)} \tag{3.6}$$

up to addition of vector fields $Y^{i_1 \ldots i_{p-1}}$ satisfying the kernel equation

$$Y^{(i_1 \ldots i_{p-1}} \rfloor \beta^{i_p)} = 0 \ . \tag{3.7}$$

The non-uniqueness can be characterized locally as follows. For $\hat{g}$ as in (3.2) the associated Hamiltonian vector fields $X^{i_1 \ldots i_{p-1}}_{\hat{g}}$ determined by equation (3.6) have the local coordinate expressions

$$\begin{aligned} X^{i_1 \ldots i_{p-1}}_{\hat{g}} &= \frac{1}{(p-1)!} (g^{j_1 \ldots j_{p1} k} \circ \pi) \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &\quad - \frac{1}{p!} \{ \frac{\partial}{\partial x^l} (g^{j_1 \ldots j_p} \circ \pi) \pi^{i_1}_{j_1} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^k_{j_p} + T^{i_1 \ldots i_{p-1} k}_l \} \frac{\partial}{\partial \pi^k_l} \ . \end{aligned} \tag{3.8}$$

The non-uniqueness is contained completely in the vertical component

$$Y^{i_1 \ldots i_{p-1}} = T^{i_1 \ldots i_{p-1} k}_j \frac{\partial}{\partial \pi^k_j} \ , \tag{3.9}$$

where the coefficients $T_j^{i_1 \ldots i_{p-1} k}$ must satisfy

$$T_j^{(i_1 \ldots i_{p-1} k)} = 0 \tag{3.10}$$

but are otherwise arbitrary.

There are special cases (see, for example, Section 7) in which the non-uniqueness can be removed globally by placing additional invariantly defined conditions on the vector fields $X_{\hat{g}}^{i_1 \ldots i_{p-1}}$ in addition to equation (3.6). Here, however, we will resolve the non-uniqueness by working locally as follows.

We assign to $\hat{g} \in ST^p$ the $N_S(p)$ Hamiltonian vector fields $X_{\hat{g}}^{i_1 \ldots i_{p-1}}$ determined by equation (3.6) and the conditions

$$d\pi_j^k \left(X_{\hat{g}}^{i_1 \ldots i_{p-1}} \right) = d\pi_j^{(k} \left(X_{\hat{g}}^{i_1 \ldots i_{p-1})} \right) \quad . \tag{3.11}$$

This condition is clearly a local condition and accordingly for the moment we restrict attention to objects defined locally on $\pi^{-1}(U) \subset LM$, where $U \subset M$ is the domain of the chart (x^i). Since the undetermined elements of $X_{\hat{g}}^{i_1 \ldots i_{p-1}}$ given in (3.8) satisfy (3.10), the auxiliary conditions (3.11) now provides uniqueness locally, with

$$\begin{aligned} X_{\hat{g}}^{i_1 \ldots i_{p-1}} &= \frac{1}{(p-1)!} (g^{j_1 \ldots j_{p1} k} \circ \pi) \pi_{j_1}^{i_1} \cdots \pi_{j_{p-1}}^{i_{p-1}} \frac{\partial}{\partial x^k} \\ &- \frac{1}{p!} \frac{\partial}{\partial x^l} (g^{j_1 \ldots j_p} \circ \pi) \pi_{j_1}^{i_1} \cdots \pi_{j_{p-1}}^{i_{p-1}} \pi_{j_p}^k \frac{\partial}{\partial \pi_l^k} \quad . \end{aligned} \tag{3.12}$$

Returning now to equation (3.3) we see that

$$\begin{aligned} \{\hat{f}, \hat{g}\}^{ij} &= X_{\hat{f}}(\hat{g}^{ij}) \\ &= d\hat{g}^{ij}(X_{\hat{f}}) \\ &= -4\beta^{(i}(X_{\hat{g}}^{j)}, X_{\hat{f}}) \\ &= +4\beta^{(i}(X_{\hat{f}}, X_{\hat{g}}^{j)}) \\ &= -2d\hat{f}^{(i}(X_{\hat{g}}^{j)}) \\ &= -2X_{\hat{g}}^{(i}(\hat{f}^{j)}) \quad . \end{aligned} \tag{3.13}$$

Thus the definition

$$\{\hat{g}, \hat{f}\} \stackrel{def}{=} 2! X_{\hat{g}}^{(i}(\hat{f}^{j)}) r_i \otimes r_j \tag{3.14}$$

has the right properties to make sense of equation (3.4).

More generally, let $\hat{f} \in \mathrm{ST}^p$ and $\hat{g} \in \mathrm{ST}^q$, and denote their corresponding Hamiltonian vector fields, determined uniquely but locally by equations (3.6) and (3.11), by $X_{\hat{f}}^{i_1 \ldots i_{p-1}}$ and $X_{\hat{g}}^{i_1 \ldots i_{q-1}}$, respectively. Then one may show that the two definitions

$$\{\hat{f}, \hat{g}\} \stackrel{def}{=} p! X_{\hat{f}}^{(i_1 \ldots i_{p-1}} (\hat{g}^{i_p \ldots i_{p+q-1})}) r_{i_1} \otimes \cdots \otimes r_{i_{p+q-1}} \tag{3.15}$$

and

$$\{\hat{g}, \hat{f}\} \stackrel{def}{=} q! X_{\hat{g}}^{(i_1 \ldots i_{q-1}} (\hat{f}^{i_q \ldots i_{q+p-1})}) r_{i_1} \otimes \cdots \otimes r_{i_{p+q-1}} \tag{3.16}$$

are related by

$$\{\hat{g}, \hat{f}\} = -\{\hat{f}, \hat{g}\} \ .$$

The fact that if $\hat{f} \in \mathrm{ST}^p$ and $\hat{g} \in \mathrm{ST}^q$, then $\{\hat{f}, \hat{g}\} \in \mathrm{ST}^{p+q-1}$ follows from the following arguement. Writing out the explicit formula for $\{\hat{f}, \hat{g}\}$ using (3.2), (3.12) and (3.15) we obtain

$$\{\hat{f}, \hat{g}\}^{i_i \ldots i_{p+q-1}} = \begin{pmatrix} p f^{k(j_2 \ldots j_p} \partial_k g^{l a_2 \ldots a_q)} \\ \\ -q g^{k(a_2 \ldots a_q} \partial_k f^{l j_2 \ldots j_p)} \end{pmatrix} \pi_l^{(i_1} \pi_{j_2}^{i_2} \cdots \pi_{j_p}^{i_p} \pi_{a_2}^{i_{p+1}} \ldots \pi_{a_q}^{i_{p+q-1})} \quad . \tag{3.17}$$

The right hand side of this equation is the $\otimes_s^{p+q-1}\mathbb{R}^n$-valued function on LM corresponding to the **differential concomitant**, due to Schouten and Nijenhuis [14,15], of the tensor fields $\vec{f}$ and $\vec{g}$ on M. It is easy to show that ∂_k on the right hand side of (3.17) can be replaced by ∇_k for an arbitrary symmetric linear connection, and hence the right hand side of (3.17) is independent of coordinates. Thus $\{\hat{f}, \hat{g}\}$ is an element of ST^{p+q-1}.

This is a rather remarkable result. It shows that although the vector-valued vector fields $X_{\hat{f}}$ given in (3.12) depend explicitly on the choice of coordinate chart, the bracket $\{\hat{f}, \hat{g}\}$ calculated explicitly in terms of $X_{\hat{f}}$ itself does not depend on the choice of coordinates. We can see why this is true by noting that

$$\begin{aligned} \{\hat{f}, \hat{g}\}^{i_1 \ldots i_{p+q-1}} &= p! X_{\hat{f}}^{(i_1 \ldots i_{p-1}} (g^{i_p \ldots i_{p+q-1})}) \\ &= -p! q! d\theta^{(i_1} (X_{\hat{g}}^{i_2 \ldots i_q}, X_{\hat{f}}^{i_{q+1} \ldots i_{p+q-1})}) \quad . \end{aligned}$$

It is clear from this last equation and (3.10) that the undetermined components $T_j^{i_1 \ldots i_{p-1} k}$ do not contribute to the bracket $\{\hat{f}, \hat{g}\}$.

For $\hat{f} \in \mathrm{ST}^p$ we get from (3.6) and (3.11) a unique set of locally defined Hamiltonian vector fields $X_{\hat{f}}^{i_1 \ldots i_{p-1}}$. We introduce multi-index notation $I = (i_1 \ldots i_{p-1})$ and view the set of vector fields $X_{\hat{f}}^{i_1 \ldots i_{p-1}}$ as the $\otimes_s^{p-1}\mathbb{R}^n$-valued vector field $\hat{X}_{\hat{f}}$ defined by

$$\begin{aligned} \hat{X}_{\hat{f}} &= X_{\hat{f}}^I \otimes r_I \quad , \\ r_I &= r_{i_1} \otimes_s r_{i_2} \otimes_s \cdots \otimes_s r_{i_{p-1}} \quad . \end{aligned} \tag{3.18}$$

For two **arbitrary** vector fields $\hat{X} = X^I \otimes r_I$ and $\hat{Y} = Y^J \otimes r_J$ with values in $\otimes_s^{p-1}\mathbb{R}^n$ and $\otimes_s^{q-1}\mathbb{R}^n$, respectively, we define a bracket by

$$[\hat{X}, \hat{Y}] = [X^I, Y^J] \otimes r_I \otimes_s r_J \quad . \tag{3.19}$$

The bracket on the right hand side of this last equation is the ordinary Lie bracket of vector fields. One shows directly that the bracket defined in (3.19) has all the properties of a Lie bracket. Thus the infinite dimensional vector space

$$S\hat{\mathcal{X}}(LM) = \sum_{p=1}^{\infty} S\hat{\mathcal{X}}^p(LM) \quad ,$$

where $S\hat{\mathcal{X}}^p(LM)$ denotes the vector space of $\otimes_s^{p-1}\mathbb{R}^n$-valued vector fields on LM, is a **Lie algebra** under the bracket defined in (3.19).

We introduce the notation

- LSHV^p= the set of all $\otimes_s^{p-1}\mathbb{R}^n$-valued Hamiltonian vector fields $\hat{X}_{\hat{f}}$ determined locally by elements $\hat{f} \in ST^p$ by equations (3.6) and (3.11),
- LSHV=$\sum_{p=1}^{\infty}$ LSHV^p

Lemma: Let $X_{\hat{g}}^J$ be the Hamiltonian vector field determined locally by $\hat{g} \in ST^q$ where J denotes the multi-index $J = i_1 i_2 \ldots i_{q-1}$. Then

$$L_{X_{\hat{g}}^{(J}} \beta^{i)} = 0$$

where the notation (J and i) indicates symmetrization over the indices.

Proof: Using the identity $L_X\omega = X \rfloor\ d\omega + d(X \rfloor\ \omega)$ we obtain

$$L_{X_g^{(J}} \beta^{i)} = X_g^{(J} \rfloor\ d\beta^{i)} + d(X_g^{(J} \rfloor\ \beta^{i)}) \quad .$$

The first term on the right hand side vanishes since $\beta^i = d\theta^i$, and the second term vanishes because by (3.8) $X_g^{(J} \rfloor\ \beta^{i)} = (-1/q!)d(\hat{g}^{Ji})$. ∎

The next two theorems will establish that the space ST of symmetric tensorial 0-forms, and the local spaces $LSHV$ of the corresponding locally defined Hamiltonian vector fields, are Lie algebras under multiplications defined by the Poisson bracket (3.15) and Lie bracket (3.19), respectively.

Theorem 3.1: Let $\hat{f} \in ST^p$ and $g \in ST^q$, and denote the corresponding locally Hamiltonian vector fields by $\hat{X}_{\hat{f}}$ and $\hat{X}_{\hat{g}}$. Then

$$[\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] = \frac{(p+q-1)!}{p!q!}\hat{X}_{\{\hat{f},\hat{g}\}} \quad .$$

Proof:
Using the identity $L_X(Y \rfloor\ \beta) = X \rfloor\ (L_Y\beta) + [X,Y] \rfloor\ \beta$ we have

$$[X^{(I}_{\hat{f}}, X^{J}_{\hat{g}}] \rfloor\ \beta^{i)} = L_{X^{(I}_{\hat{f}}}(X^{J}_{\hat{g}} \rfloor\ \beta^{i)} - X^{(I}_{\hat{f}} \rfloor\ (L_{X^{J}_{\hat{g}}}\beta^{i)}$$

with symmetrization over the indices I, J, i. The second term on the right hand side of this equation vanishes by the lemma. Using the identity $L_X\omega = X \rfloor\ d\omega + d(X \rfloor\ \omega)$ the equation now reduces to

$$\begin{aligned}
[X^{(I}_{\hat{f}}, X^{J}_{\hat{g}}] \rfloor\ \beta^{i)} &= X^{(I}_{\hat{f}} \rfloor\ d(X^{J}_{\hat{g}} \rfloor\ \beta^{i)}) + d(X^{(I}_{\hat{f}} \rfloor\ X^{J}_{\hat{g}} \rfloor\ \beta^{i)}) \\
&= d(X^{(I}_{\hat{f}} \rfloor\ X^{J}_{\hat{g}} \rfloor\ \beta^{i)})
\end{aligned}$$

where the second line follows from the lemma. If we now use equation (3.6) and the definition (3.15) we obtain

$$\begin{aligned}
[X^{(I}_{\hat{f}}, X^{J}_{\hat{g}}] \rfloor\ \beta^{i)} &= d(X^{(I}_{\hat{f}} \rfloor\ X^{J}_{\hat{g}} \rfloor\ \beta^{i)}) \\
&= d(X^{(I}_{\hat{f}} \rfloor\ (\frac{-1}{q!}d\hat{g}^{Ji)})) \\
&= \frac{-1}{q!}d(X^{(I}_{\hat{f}}(\hat{g}^{Ji)})) \\
&= -d(\frac{1}{p!q!}\{\hat{f},\hat{g}\}^{(IJi)}) \quad .
\end{aligned}$$

The result now follows from equation (3.6). ▮

Since the bracket used in this theorem is a (generalized) Lie bracket of vector-valued vector fields we have the

Corollary 3.1: The vector spaces of locally defined Hamiltonian vector fields LSHV determined by elements of ST are **Lie algebras** under the bracket defined in (3.19).

The formula

$$\{\hat{f}, \hat{g}\}^{iIJ} = 2p!q!\beta^{(i}(X^I_g, X^{J)}_g) \quad , \tag{3.20}$$

which follows easily from (3.6) and (3.15), will be useful in the following.

Theorem 3.2: The bracket defined in (3.15) for elements of ST satisfies the Jacobi identity. That is, for $\hat{f} \in ST^p$, $\hat{g} \in ST^q$ and $\hat{h} \in ST^r$

$$\{\hat{f}, \{\hat{g}, \hat{h}\}\} + \{\hat{h}, \{\hat{f}, \hat{g}\}\} + \{\hat{g}, \{\hat{h}, \hat{f}\}\} = 0 \quad .$$

Proof: Let X_f^I, X_g^J and X_h^K denote the Hamiltonian vector fields determined by $\hat{f}$, $\hat{g}$ and $\hat{h}$, where I, J, K denote the multi-indices $I = i_2 i_3 \dots i_p$, $J = i_{p+1} i_{p+2} \dots i_{p+q-1}$ and $K = i_{p+q} i_{p+q+1} \dots i_{p+q+r-2}$. Then by using the standard identity for evaluating $d\omega(X, Y, Z)$ for ω a 2-form we obtain

$$\begin{aligned}
0 &= 3d\beta^{(i_1}(X_{\hat{f}}^I, X_{\hat{g}}^J, X_{\hat{h}}^{K)}) \\
&= X_{\hat{f}}^{(I}\beta^{i_1}(X_{\hat{g}}^J, X_{\hat{h}}^{K)}) + X_{\hat{g}}^{(J}\beta^{i_1}(X_{\hat{h}}^K, X_{\hat{f}}^{I)}) + X_{\hat{h}}^{(K}\beta^{i_1}(X_{\hat{f}}^I, X_{\hat{g}}^{J)}) \\
&- \beta^{(i_1}([X_{\hat{f}}^I, X_{\hat{g}}^J], X_{\hat{h}}^{K)}) - \beta^{(i_1}([X_{\hat{h}}^K, X_{\hat{f}}^I], X_{\hat{g}}^{J)}) - \beta^{(i_1}([X_{\hat{g}}^J, X_{\hat{h}}^K], X_{\hat{f}}^{I)}) \quad .
\end{aligned}$$

Using the formula (3.20) and Theorem 3.1 in this equation we obtain

$$\begin{aligned}
0 &= X_{\hat{f}}^{(I}(\frac{1}{2q!r!}\{\hat{g}, \hat{h}\}^{JKi_1)}) + X_{\hat{h}}^{(J}(\frac{1}{2p!q!}\{\hat{f}, \hat{g}\}^{IJi_1)}) \\
&+ X_{\hat{g}}^{(K}(\frac{1}{2p!r!}\{\hat{h}, \hat{f}\}^{KIi_1)}) - \beta^{(i_1}(\frac{(p+q-1)!}{p!q!}X_{\{\hat{f},\hat{g}\}}^{IJ}, X_h^{K)}) \\
&- \beta^{(i_1}(\frac{(p+r-1)!}{p!r!}X_{\{\hat{h},\hat{f}\}}^{KI}, X_g^{J)}) - \beta^{(i_1}(\frac{(q+r-1)!}{q!r!}X_{\{\hat{g},\hat{h}\}}^{JK}, X_f^{I)}) \quad .
\end{aligned}$$

Next we use the definition (3.15) in the first three terms and formula (3.18) and Theorem 3.1 in the last three terms to obtain

$$\begin{aligned}
0 &= \frac{1}{2p!q!r!}\{\hat{f}, \{\hat{g}, \hat{h}\}\}^L + \frac{1}{2p!q!r!}\{\hat{h}, \{\hat{f}, \hat{g}\}\}^L + \frac{1}{2p!q!r!}\{\hat{g}, \{\hat{h}, \hat{f}\}\}^L \\
&- \frac{(p+q-1)!}{p!q!}(\frac{1}{2(p+q-1)!r!}\{\{\hat{f}, \hat{g}\}, \hat{h}\}^L) \\
&- \frac{(p+r-1)!}{p!r!}(\frac{1}{2(p+r-1)!q!}\{\{\hat{h}, \hat{f}\}, \hat{g}\}^L) \\
&- \frac{(q+r-1)!}{q!r!}(\frac{1}{2(q+r-1)!p!}\{\{\hat{g}, \hat{h}\}, \hat{f}\}^L) \quad ,
\end{aligned}$$

where the multi-index L denotes $(i_1 IJK)$. Cancelling the common factor $\frac{1}{p!q!r!}$ we obtain

$$
\begin{aligned}
0 &= \frac{1}{2}\{\hat{f},\{\hat{g},\hat{h}\}\}^L + \frac{1}{2}\{\hat{h},\{\hat{f},\hat{g}\}\}^L \\
&+ \frac{1}{2}\{\hat{g},\{\hat{h},\hat{f}\}\}^L - \frac{1}{2}\{\{\hat{f},\hat{g}\},\hat{h}\}^L \\
&- \frac{1}{2}\{\{\hat{h},\hat{f}\},\hat{g}\}^L - \frac{1}{2}\{\{\hat{g},\hat{h}\},\hat{f}\}^L \\
&= \{\hat{f},\{\hat{g},\hat{h}\}\}^L + \{\hat{h},\{\hat{f},\hat{g}\}\}^L + \{\hat{g},\{\hat{h},\hat{f}\}\}^L \quad . \qquad \blacksquare
\end{aligned}
$$

The symmetrized tensor product $\otimes_s$ makes ST into a commutative algebra. If we now consider again elements $\hat{f} \in ST^p$, $\hat{g} \in ST^q$ and $\hat{h} \in ST^r$, then by using definition (3.15) and techniques as in the proofs above one may show that

$$
\{\hat{f}, \hat{g} \otimes_s \hat{h}\} = \{\hat{f}, \hat{g}\} \otimes_s \hat{h} + \hat{g} \otimes_s \{\hat{f}, \hat{h}\} \quad . \tag{3.21}
$$

Thus the bracket defined in (3.15) acts as a derivation on the commutative algebra, and as a result we have

Theorem 3.3: The space ST of symmetric tensorial 0-forms on LM is a **Poisson algebra** with respect to the Poisson bracket defined in (3.15).

4. The Poisson Super Algebra AT on LM

The analysis presented in Section 3 can be modified to define a symplectic geometry for anti-symmetric contravariant tensor fields. We will find that the natural generalization of the Poisson bracket to the space $AT = \sum_{p=1}^{\infty} AT^p$ of anti-symmetric $\otimes_a^p\mathbb{R}^n$-valued tensorial functions on LM will make AT into a Poisson super algebra, while the corresponding sets of locally defined vector-valued Hamiltonian vector fields will form super algebras. Because of a uniqueness problem of the type encountered in Section 3 the Hamiltonian vector fields will be defined locally on $\pi^{-1}(U)$ where (x^i, U) is a chart domain on M.

To $\hat{f} \in AT^p$ one assigns an anti-symmetric $\otimes_a^p\mathbb{R}^n$-valued Hamiltonian vector field

$$
\hat{X}_{\hat{f}} = X_{\hat{f}}^{i_1 i_2 \ldots i_{p-1}} r_{i_1} \otimes_a \ldots \otimes_a r_{i_{p-1}} := X_{\hat{f}}^I \otimes r_{[I]} \quad ,
$$

where the multi-index I is defined by $I = i_1 i_2 \ldots i_{p1}$, and where $r_{[I]} := r_1 \otimes_a r_2 \otimes_a \ldots \otimes_a r_{p1}$. Here square brackets on indices denotes anti-symmetrization. We will refer to $\hat{X}_{\hat{f}}$ with values in $\otimes_a^{p-1}\mathbb{R}^n$ as a **rank p** Hamiltonian vector field. The $N_A(p) = \begin{pmatrix} n \\ p-1 \end{pmatrix}$ component vector fields $X_{\hat{f}}^{[i_1 i_2 \ldots i_{p-1}]}$ are determined by the generalized symplectic structure equation

$$d\hat{f}^{i_1\ldots i_p} = -p! X_f^{[i_1\ldots i_{p-1}} \lrcorner\ \beta^{i_p]} \quad . \tag{4.1}$$

The map $\hat{f} \longrightarrow \hat{X}_f$ is also not unique because of the anti-symmetrization in (4.1). The non-uniqueness is also contained completely in a vertical component $T_j^{i_1 i_2 \ldots i_{p-1} k} \frac{\partial}{\partial \pi_j^k}$ that is arbitrary except for the condition

$$T_j^{[i_1 i_2 \ldots i_{p-1} k]} = 0 \quad . \tag{4.2}$$

As in Section 3 we can therefore determine $\hat{X}_{\hat{f}}$ uniquely, given $\hat{f}$, by equation (4.1) and the locally defined auxillary condition

$$d\pi_j^k(X_{\hat{f}}^{i_1 i_2 \ldots i_{p-1}}) = d\pi_j^{[k}(X_{\hat{f}}^{i_1 i_2 \ldots i_{p-1}]}) \quad . \tag{4.3}$$

The explicit local formula for $X_{\hat{f}}$ determined by $\hat{f} \in AT^p$ from (4.1) and (4.3) is

$$\begin{aligned} X_{\hat{g}}^{i_1\ldots i_{p-1}} = & \frac{1}{(p-1)!}(g^{j_1\ldots j_{p1}k}\circ\pi)\pi_{j_1}^{i_1}\cdots\pi_{j_{p-1}}^{i_{p-1}}\frac{\partial}{\partial x^k} \\ & -\frac{1}{p!}\frac{\partial}{\partial x^l}(g^{j_1\ldots j_p}\circ\pi)\pi_{j_1}^{i_1}\cdots\pi_{j_{p-1}}^{i_{p-1}}\pi_{j_p}^k\frac{\partial}{\partial\pi_l^k} \quad . \end{aligned} \tag{4.4}$$

Now let $LAHV = \sum_{p=1}^{\infty} LAHV^p$, where $LAHV^p$ denotes the vector space of locally defined $\otimes_a^{p-1}\mathbb{R}^n$-valued vector fields determined uniquely by elements of AT^p from equations (4.1) and (4.3). Using these Hamiltonian vector fields we define a map $\{,\}$: $AT^p \times AT^q \longrightarrow AT^{p+q-1}$ by

$$\{\hat{f}, \hat{g}\}^{i_1\ldots i_{p+q-1}} = p! X_{\hat{f}}^{[i_1\ldots i_{p-1}}(\hat{g}^{i_p\ldots i_{p+q-1}]}) \quad . \tag{4.5}$$

Working out the right hand side of this equation using (4.4) we obtain

$$\{\hat{f}, \hat{g}\}^{i_1\ldots i_{p+q-1}} = \begin{pmatrix} p f^{k[j_2\ldots j_p}\partial_k g^{l a_2\ldots a_q]} \\ \\ -q g^{k[a_2\ldots a_q}\partial_k f^{l j_2 \ldots j_p]} \end{pmatrix} \pi_l^{[i_1}\pi_{j_2}^{i_2}\cdots\pi_{j_p}^{i_p}\pi_{a_2}^{i_{p+1}}\ldots\pi_{a_q}^{i_{p+q-1}]} \quad . \tag{4.6}$$

The right hand side of this equation is the $\otimes_a^{p+q-1}\mathbb{R}^n$-valued function on LM corresponding to the differential concomitant, due to Schouten and Nijenhuis [14,15], of the anti-symmetric tensor fields $\vec{f}$ and $\vec{g}$ on M. It is easy to show that ∂_k on the right hand side of

(4.6) can be replaced by ∇_k for an arbitrary symmetric linear connection, and hence the right hand side of (4.6) is independent of coordinates. Thus for $\hat{f} \in AT^p$ and $\hat{g} \in AT^q$ the bracket $\{\hat{f}, \hat{g}\}$ is an element of AT^{p+q-1}.

We also define a bracket $[\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}]$ for Hamiltonian vector fields $\hat{X}_{\hat{f}}$ by

$$[\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] := [X^I_{\hat{f}}, X^J_{\hat{g}}] \otimes r_{[IJ]} \quad . \tag{4.7}$$

As we will see the brackets in (4.5) and (4.7) are not Lie brackets, but rather are brackets appropriate for super-algebras.

The following theorems establish basic facts about the spaces AT and $LAHV$. The proofs are omitted since most are simply modifications of the corresponding proofs given in Section 3.

Theorem 4.1 For all $\hat{f} \in AT^p$, $\hat{g} \in AT^q$, and $\hat{h} \in AT^r$, the bracket defined in (4.5) has the following properties:

$$\begin{aligned}
&(a)\quad \{\hat{f}, \hat{g}\} = -(-1)^{(p-1)(q-1)}\{\hat{g}, \hat{f}\} \\
&(b)\quad 0 = (-1)^{(p-1)(r-1)}\{\hat{f}, \{\hat{g}, \hat{h}\}\} + (-1)^{(p-1)(q-1)}\{\hat{g}, \{\hat{h}, \hat{f}\}\} + (-1)^{(q-1)(r-1)}\{\hat{h}, \{\hat{f}, \hat{g}\}\} \\
&(c)\quad \{\hat{f}, \hat{g} \wedge \hat{h}\} = \{\hat{f}, \hat{g}\} \wedge \hat{h} + (-1)^{(p-1)q}\hat{f} \wedge \{\hat{g}, \hat{h}\} \quad .
\end{aligned} \tag{4.8}$$

An algebra with properties (a) and (b) above has been given the name **super-algebra** by physicists [17,18]. When the derivation property (c) is included the resulting algebra has been given the name **Schouten algebra** [11]. In keeping with the general philosophy of generalizing standard symplectic geometry we will refer to the algebra AT as a **Poisson super algebra**.

Theorem 4.2 Let $\hat{X}_{\hat{f}}$, $\hat{X}_{\hat{g}}$ and $\hat{X}_{\hat{h}}$ be elements of LHV of rank p, q and r, respectively. Then under the bracket defined in (4.5) above

$$\begin{aligned}
&(a)\quad [\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] = \frac{(p+q-1)!}{p!q!}\hat{X}_{\{\hat{f},\hat{g}\}} \quad , \\
&(b)\quad [\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] = -(-1)^{(p-1)(q-1)}[\hat{X}_{\hat{g}}, \hat{X}_{\hat{f}}] \quad , \\
&(c)\quad 0 = (-1)^{(p-1)(r-1)}[\hat{X}_{\hat{f}}, [\hat{X}_{\hat{g}}, \hat{X}_{\hat{h}}]] + (-1)^{(p-1)(q-1)}[\hat{X}_{\hat{g}}, [\hat{X}_{\hat{h}}, \hat{X}_{\hat{f}}]] \\
&\qquad + (-1)^{(q-1)(r-1)}[\hat{X}_{\hat{h}}, [\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}]] \ .
\end{aligned}$$

Thus each local algebra $(LHV, [\ ,\ ])$, with bracket for $\otimes_a^{p-1}\mathbb{R}^n$-valued vector fields defined as in (4.5) above, is a **local super algebra**.

5. Geometrical Applications

In this section we consider two applications of the algebras introduced in Sections 3 and 4. As a first example we show how to define the natural lift of a symmetric or anti-symmetric contravariant tensor field on M to LM. As discussed in Section 2 the Hamiltonian vector field $X_{\hat{f}}$ determined by $\hat{f} \in \mathrm{ST}^1$ is the natural lift to LM, as defined for example in [16], of the corresponding vector field $\vec{f}$ on M . What has been shown in Section 2 is that the definition of natural lift of a vector field to LM, in terms of the three properties given in equation (2.20), can be replaced by the

DEFINITION: The natural lift of a vector field $\vec{f}$ on M to LM is the Hamiltonian vector field $X_{\hat{f}}$ determined by the equation

$$d\hat{f} = -X_{\hat{f}} \lrcorner\ \beta$$

where $\hat{f}$ is the element of ST^1 determined by $\vec{f}$.

With this fact in hand it is natural to use the symplectic structure to extend the definition of natural lift to arbitrary symmetric contravariant tensor fields. We illustrate the extension for a rank 2 tensor field. Thus let $\hat{g} \in \mathrm{ST}^2$ corresponding to the rank 2 tensor field $\vec{g}$ on M. Then the Hamiltonian vector fields $X^i_{\hat{g}}$ determined locally by $\hat{g}$ are (from equation (3.8))

$$X^i_{\hat{g}} = (g^{ab} \circ \pi)\pi^i_a \frac{\partial}{\partial x^b} - \frac{1}{2}\{\frac{\partial}{\partial x^j}(g^{ab} \circ \pi)\pi^i_a \pi^k_b \frac{\partial}{\partial \pi^k_j}\} \quad . \tag{5.1}$$

It is a simple matter to check that the vector fields $X^i_{\hat{g}}$ have, instead of property (1) listed in equation (2.20), the transformation property

$$dR_a(X^i_{\hat{g}}) = (a^{-1})^i_j X^j_{\hat{g}} \quad \forall\ a \in GL(n) \quad .$$

Recalling definition (2.2) we define a map

$$u \longrightarrow d_u\pi \otimes u : T_u(LM) \otimes \mathbb{R}^n \longrightarrow T_{\pi(u)}M \otimes T_{\pi(u)}M$$

by

$$(d_u\pi \otimes u)(X \otimes \eta) \stackrel{def}{=} d_u\pi(X) \otimes u(\eta) \qquad \text{for} \quad X \in \mathrm{T}_u LM \ , \ \eta \in \mathbb{R}^n \ . \tag{5.2}$$

It is then easy to check that if $\hat{X}_{\hat{g}} = X^i_{\hat{g}} \otimes r_i$, with the $X^i_{\hat{g}}$ as given above, then

$$d_u\pi \otimes u(\hat{X}_{\hat{g}}) = \vec{g}(\pi(u)) \ . \tag{5.3}$$

Two remarks are in order. First it is clear that the domain $T_uLM \otimes \mathbb{R}^n$ in the above may be replaced by $T_uLM \otimes (\otimes^p\mathbb{R}^n)$ by using (2.2) on each factor of $\otimes^p\mathbb{R}^n$. Second, we note that although the vector-valued vector fields $\hat{X}_{\hat{g}}$ are not uniquely determined for $p > 1$, the non-unique components are, by (3.9), vertical and hence do not contribute to the right hand side of (5.2). It is therefore natural to make the following

DEFINITION: The natural lift of a symmetric contravariant rank p tensor field $\vec{f}$ on M to LM is the symmetric $\otimes^{p1}_s\mathbb{R}^n$-valued vector field

$$\hat{X}_{\hat{f}} = X^{i_1\ldots i_{p-1}}_{\hat{f}} r_{i_1} \otimes_s \cdots \otimes_s r_{i_{p-1}}$$

where the component vector fields $X^{i_1\ldots i_{p-1}}_{\hat{f}}$ are determined locally by equations (3.6) and (3.11).

It is clear that this definition may be modified to give a definition of the natural lift of an anti-symmetric contravariant tensor field $\vec{f}$ on M to LM as the corresponding $\otimes^{p-1}_a\mathbb{R}^n$-valued vector field $\hat{X}_{\hat{f}}$.

As a second geometrical application we consider the definition of Killing tensors. Suppose that $\vec{g}$ is a contravariant metric tensor field on a manifold M, and denote by $\hat{g}$ the corresponding element of ST^2. Then we may define p^{th} **order Killing tensors** as elements $\hat{K} \in \mathrm{ST}^p$ that **commute with** $\hat{g}$ **under the Poisson bracket**. The analogy from mechanics is that if we consider $\hat{g}$ as a generalized Hamiltonian (see Section 7), then the equation $\{\hat{K}, \hat{g}\} = 0$ is the **constants of the motion** equation.

For $p = 1$ we have the well known definition of Killing vector fields since, as argued in Section 2, for $\hat{K} \in ST^1$ and $\hat{g} \in ST^2$ the Poisson bracket $\{\hat{K}, \hat{g}\}$ corresponds to the Lie derivative of $\vec{g}$ with respect to $\vec{K}$.

For $p = 2$ the vanishing of the bracket $\{\hat{K}, \hat{g}\}$ can easily be shown to reduce, using (3.17), to the equation

$$\nabla^{(i}(K^{jk)}) = 0 \quad , \tag{5.4}$$

where covariant differentiation is with respect to the Levi-Civita connection defined by $\vec{g}$. This definition (5.4) of rank 2 Killing tensors has been used, for example, by Sommers [19].

6. Local Spaces of Allowable Observables on LM

As remarked in Section 2, not all $\otimes_s^p \mathbb{R}^n$-valued functions on LM are compatible with the equation

$$d\hat{g}^{i_1 \ldots i_p} = -p! X_{\hat{g}}^{(i_1 \ldots i_{p-1}} \rfloor \beta^{i_p)} \quad .$$

In order to find the general form of allowable observables on LM we proceed as follows. Generalizing a definition from standard symplectic geometry we have

DEFINITION: A $\otimes_s^{p-1} \mathbb{R}^n$-valued vector field $X^I \otimes r_I$ on LM is **locally Hamiltonian** if

$$d\left(X^{(i_1 \ldots i_{p-1}} \rfloor \beta^{i_p)}\right) = 0 \quad . \tag{6.1}$$

The corresponding condition $d(X \rfloor \omega) = 0$ on T^*M implies that locally on $U \subset T^*M$ there is a function $f : U \to \mathbb{R}$ such that $X = X_f$, but places no further restrictions on f. On the other hand equation (6.1) asserts that (A) there exists on $\hat{U} \subset LM$ a function $\hat{g} : \hat{U} \to \otimes_s^p \mathbb{R}^n$ such that $X = X_{\hat{g}}$, and (B) that $\hat{g}$ must be a polynomial of degree p in the momentum coordinates π^i_j with coefficients in the set of real-valued functions on $\pi(\hat{U}) \subset M$.

To see this consider (6.1) for the case p=2. With the vector fields X^i expressed in local coordinates as

$$X^i = X^{ij} \frac{\partial}{\partial x^j} + X^{ij}_k \frac{\partial}{\partial \pi^j_k} \tag{6.2}$$

equation (6.1) splits up into the three sets of equations

$$\frac{\partial X^{(ij)}_b}{\partial x^a} - \frac{\partial X^{(ij)}_a}{\partial x^b} = 0 \quad , \tag{6.3}$$

$$\delta^{(i}_k \frac{\partial X^{j)b}}{\partial \pi^r_s} - \delta^{(i}_r \frac{\partial X^{j)s}}{\partial \pi^k_b} = 0 \quad , \tag{6.4}$$

and

$$\frac{\partial X^{(ij)}_a}{\partial \pi^r_s} + \delta^{(i}_r \frac{\partial X^{j)s}}{\partial x^a} = 0 \quad . \tag{6.5}$$

Working first with the set of equations (6.4) one shows by a series of contractions and resubstitutions that

$$\frac{\partial X^{ib}}{\partial \pi^r_s} = \frac{1}{n} \frac{\partial X^{jb}}{\partial \pi^j_s} \delta^i_r \quad . \tag{6.6}$$

Computing a second derivative of this last equation with respect to π^a_b and resubstituting (6.6) on the right hand side after permuting the order of differentiation, one can show by contraction of the resulting equation that

$$\frac{\partial^2 X^{ij}}{\partial \pi^a_b \pi^r_s} = 0 \quad . \tag{6.7}$$

Hence the components X^{ij} are linear in the momentum coordinates π^i_j. We conclude that

$$X^{ij} = A^{jk}(x)\pi^i_k + B^{ij}(x) \quad , \; A^{ij} = A^{ji} \quad . \tag{6.8}$$

The symmetry of the coefficients A^{ij} follows from (6.4).

Now using (6.8) in the right hand side of (6.5) one can show that

$$X^{(ij)}_a = \frac{\partial}{\partial x^a}\left(\frac{-1}{2}(A^{kl}(x)\pi^{(i}_k\pi^{j)}_l + 2\pi^{(i}_k B^{j)}_k(x))\right) + C^{ij}_a(x) \quad . \tag{6.9}$$

Finally, using (6.3) and (6.9) one concludes that

$$C^{ij}_a(x) = \frac{\partial C^{ij}}{\partial x^a} \quad , \; C^{ij} = C^{ji} \quad . \tag{6.10}$$

Hence the vector fields X^i must be of the form

$$\begin{aligned} X^i = &\left(A^{jk}(x)\pi^i_k + B^{ij}(x)\right)\frac{\partial}{\partial x^j} \\ &+ \left(\frac{\partial}{\partial x^a}\left(\frac{-1}{2}(A^{kl}(x)\pi^{(i}_k\pi^{j)}_l + 2\pi^{(i}_k B^{j)k}(x) + 2C^{ij)}(x)\right)\right)\frac{\partial}{\partial \pi^j_a} \quad . \end{aligned} \tag{6.11}$$

If we now write $d\hat{g}^{ij} = -2X^{(i} \rfloor \beta^{j)}$ with $\hat{g}^{ij} = \hat{g}^{ji}$, then we find

$$\hat{g}^{ij} = A^{kl}(x)\pi^i_k\pi^j_l + \pi^{(i}_k B^{j)k}(x) + C^{ij}(x) \quad . \tag{6.12}$$

The following theorem is the general result that one can prove using methods patterned after those in the above discussion.

Theorem 6.1: If $\hat{X} = X^I \otimes r_I$ is a $\otimes_s^{p-1}\mathbb{R}^n$-valued vector field on LM satisfying

$$d\left(X^{(i_1 i_2 \ldots i_{p-1}} \rfloor \ \beta^{i_p)}\right) = 0 \quad , \tag{6.13}$$

then locally there exist $\otimes_s^p\mathbb{R}^n$-valued functions $\hat{g}$ such that $\hat{X} = \hat{X}_{\hat{g}}$ and

$$\begin{aligned}\hat{g}^{i_1 i_2 \ldots i_p} &= \frac{1}{p}\pi_{l_1}^{(i_1} \cdots \pi_{l_p}^{i_p)} A^{l_1 \ldots l_p}(x) + \frac{1}{p-1}\pi_{L_{p-2}}^{J_{p-2}}\pi_{l_p}^{(i_p} B_{1,J_{p-2}}^{i_1 \ldots i_{p-1})l_p L_{p-2}}(x) \\ &+ \frac{1}{p-2}\pi_{L_{p-3}}^{J_{p-3}}\pi_{l_p}^{(i_p} B_{2,J_{p-3}}^{i_1 \ldots i_{p-1})l_p L_{p-3}}(x) + \ldots \\ &+ \ldots + \pi_{l_p}^{(i_p} B_{p-1}^{i_1 \ldots i_{p-1})l_p}(x) + B_p^{(i_1 \ldots i_p)}(x)\end{aligned} \tag{6.14}$$

where $L_{p-k} = l_1 \ldots l_{p-k}$, $J_{p-k} = j_1 \ldots j_{p-k}$ and where $\pi_{L_{p-k}}^{J_{p-k}} = \pi_{l_1}^{j_1}\pi_{l_2}^{j_2} \cdots \pi_{l_{p-k}}^{j_{p-k}}$.

The analogous result that one can prove for anti-symmetric $\otimes_a^{p-1}\mathbb{R}^n$-valued locally Hamiltonian vector fields is:

Theorem 6.2: If $\hat{X} = X^I \otimes r_I$ is a $\otimes_a^{p-1}\mathbb{R}^n$-valued vector field on LM satisfying

$$d\left(X^{[i_1 i_2 \ldots i_{p-1}} \rfloor \ \beta^{i_p]}\right) = 0 \quad , \tag{6.15}$$

then locally there exist $\otimes_a^p\mathbb{R}^n$-valued functions $\hat{g}$ such that $\hat{X} = \hat{X}_{\hat{g}}$ and

$$\begin{aligned}\hat{g}^{i_1 i_2 \ldots i_p} &= \frac{1}{p}\pi_{l_1}^{[i_1} \cdots \pi_{l_p}^{i_p]} A^{l_1 \ldots l_p}(x) + \frac{1}{p-1}\pi_{L_{p-2}}^{J_{p-2}}\pi_{l_p}^{[i_p} B_{1,J_{p-2}}^{i_1 \ldots i_{p-1}]l_p L_{p-2}}(x) \\ &+ \frac{1}{p-2}\pi_{L_{p-3}}^{J_{p-3}}\pi_{l_p}^{[i_p} B_{2,J_{p-3}}^{i_1 \ldots i_{p-1}]l_p L_{p-3}}(x) + \ldots \\ &+ \ldots + \pi_{l_p}^{[i_p} B_{p-1}^{i_1 \ldots i_{p-1}]l_p}(x) + B_p^{[i_1 \ldots i_p]}(x)\end{aligned} \tag{6.16}$$

where $L_{p-k} = l_1 \ldots l_{p-k}$, $J_{p-k} = j_1 \ldots j_{p-k}$ and where $\pi_{L_{p-k}}^{J_{p-k}} = \pi_{l_1}^{j_1}\pi_{l_2}^{j_2} \cdots \pi_{l_{p-k}}^{j_{p-k}}$.

7. The Metric Tensor as a Generalized Hamiltonian Tensor for Free Inertial Observers

A metric tensor field $\vec{g}$ on a spacetime M defines a real-valued function $\tilde{g}$ in canonical coordinates (x^i, π_j) on T^*M by $\tilde{g}(x,\pi) = g^{ij}(x)\pi_i\pi_j$. The free-particle Hamiltonian on T^*M for this spacetime is then $\mathcal{H} = \frac{1}{2}\tilde{g}$, and the solutions of the associated Hamilton equations on T^*M are the linear geodesics of the unique Levi-Civita connection Γ_g defined by $\vec{g}$. One may then build up parallel transport of linear frames in terms of Γ_g, but geometrical ideas of this type are only indirect consequences of the Hamiltonian dynamics of $\tilde{g}$ on T^*M. We now show that the generalized Hamiltonian dynamics of $\hat{g}$ on LM gives the full Levi-Civita connection geometry directly and explicitly.

Let $\vec{g} = g^{ij}\partial_i \otimes \partial_j$ be the local coordinate form of the metric tensor on spacetime, and let $\hat{g} = (g^{ij} \circ \pi)\pi^a_i \pi^b_j r_a \otimes r_b$ denote the corresponding tensorial function in ST^2 on LM. Then from equation (3.8) the associated Hamiltonian vector fields $X^i_{\hat{g}}$ determined by equation (3.5) have the local expressions

$$X^i_{\hat{g}} = (g^{ab} \circ \pi)\pi^i_a \frac{\partial}{\partial x^b} - \frac{1}{2}\{\frac{\partial (g^{ab} \circ \pi)}{\partial x^j}\pi^i_a \pi^k_b + T^{ik}_j\}\frac{\partial}{\partial \pi^k_j} \ . \tag{7.1}$$

It is not difficult to show that if for the arbitrary functions T^{ik}_j we take smooth functions that transform under right translations on LM according to the law

$$T^{ik}_j(u \cdot h) = (h^{-1})^i_m T^{mk}_j(u) \quad , \quad \forall h \in \ GL(n) \ , \tag{7.2}$$

then the distribution Δ on LM spanned by the vector fields

$$B_i \stackrel{def}{=} \hat{g}_{ij} X^j_{\hat{g}} \tag{7.3}$$

defines a linear connection on LM . Here $\hat{g}_{ij} = (g_{ab} \circ \pi)(\pi^a_i)^{-1}(\pi^b_j)^{-1}$ where the functions g_{ij} are the components of the matrix inverse of (g^{ij}). Roughly, from equations (7.1),(7.3) and the non-singularity of $\vec{g}$, one checks that the vector fields B_i never vanish on LM and form a complement to the vertical subspace of $T_u LM$ at each $u \in LM$. The condition given in equation (7.2) is then sufficient to guarantee that the smooth distribution Δ spanned by the vector fields B_i is invariant by right translation. These properties taken together show that Δ satisfies the distributional definition [16] of a linear connection on LM.

From the method used to derive equation (3.5) ("Find a linear connection that leaves $\vec{g}$ covariant constant and ...") it is clear that the set of all connections defined in this way contains the set of "metric linear connections" defined by $\vec{g}$. Contained in this set is the Levi-Civita connection, i.e. the unique torsion–free metric linear connection defined by $\vec{g}$. This unique connection can be defined in terms of the generalized symplectic structure as follows. Require the generalized Hamiltonian vector fields to satisfy, in addition to

$$(a) \qquad d\hat{g}^{ij} = -2X^{(i}_{\hat{g}} \lrcorner \ \beta^{j)} \ , \tag{7.4}$$

the invariantly defined constraint equations

$$(b) \qquad \beta^i \lrcorner \ X^j_{\hat{g}} \lrcorner \ X^k_{\hat{g}} = 0 \quad \forall \ i,j,k = 1,\ldots,4 \ . \tag{7.5}$$

These last equations are in fact just the "torsion free" condition in a different form. Equations (7.4) and (7.5) uniquely determine the arbitrary functions T^{ij}_k so that the resulting Hamiltonian vector fields are

$$X^i_{\hat{g}} = (g^{ab} \circ \pi)\pi^i_a \frac{\partial}{\partial x^b} + (\Gamma^b_{jc} \circ \pi) g^{ac} \pi^i_a \pi^k_b \frac{\partial}{\partial \pi^k_j} \ . \tag{7.6}$$

The functions Γ^b_{jc} are the Christoffel symbols of the Levi-Civita connection defined by $\vec{g}$. In the following it will be convenient to drop the composition "$\circ\, \pi$" whenever there is no possibility of confusion.

It is straight forward to check that the distribution spanned by the vector fields

$$\begin{aligned} B_k &= \hat{g}_{ki} X^i_g \\ &= (\pi^j_k)^{-1} \left(\frac{\partial}{\partial x^j} + \Gamma^a_{ij} \pi^b_a \frac{\partial}{\partial \pi^b_i} \right) \end{aligned} \tag{7.7}$$

is the horizontal distribution of the Levi-Civita connection. The vector fields B_i are easily seen to be the "standard horizontal vector fields" [16] determined by the connection.

For simplicity we work with the Hamiltonian vector fields $X^i_{\hat{g}}$ defined in equation (7.6). The first question we ask is: **What dynamics is determined by the four Hamiltonian vector fields** $X^i_{\hat{g}}$? When there is only a single Hamiltonian vector field $X_{\hat{f}}$, as in the case for $\hat{f} \in T^1$ as well as in standard symplectic geometry on T^*M, then the dynamics is given by the integral curves of $X_{\hat{f}}$. One can ask if the distribution spanned by the $X^i_{\hat{g}}$ is integrable, but it is well-known that only flat connections have integrable distributions. On the other hand the vector fields B_k, and hence also the vector fields $X^i_{\hat{g}}$, are tangent to the subbundle of orthonormal linear frames $O_{\hat{g}}(M)$ determined by $\hat{g}$. Thus we may define an "integral" of the set of Hamiltonian vector fields $X^i_{\hat{g}}$ to be $O_{\hat{g}}(M)$. The subbundle $O_{\hat{g}}(M)$ is thus the analogue of the "constant energy surfaces" in standard symplectic geometry.

Since a section of $O_{\hat{g}}(M)$ represents a local orthonormal linear frame field on M we conclude that *the dynamics defined by the four Hamiltonian vector fields is the dynamics of orthonormal frames, and hence the dynamics of local observers on spacetime*. More explicitly, consider the integral curves of the Hamiltonian vector field $X^1_{\hat{g}}$ with "time-like" initial conditions, i.e. $u = (p, e_i) \in LM$ with e_1 a time-like vector in T_pM. The differential equations for the integral curve of $X^1_{\hat{g}}$ through u are, from equation (7.6),

$$\begin{aligned} &(a) \qquad \frac{dx^i}{dt} = g^{ij}\pi^1_j \ , \\ &(b) \qquad \frac{d\pi^k_j}{dt} = \Gamma^b_{jc} g^{ac} \pi^1_a \pi^k_b \ . \end{aligned} \tag{7.8}$$

These two equations decouple into two sets of equations. For k=1 we obtain

$$\begin{aligned} (a') \qquad & \frac{dx^i}{dt} = g^{ij}\pi^1_j \ , \\ (b') \qquad & \frac{d\pi^1_j}{dt} = \Gamma^b_{jc} g^{ac}\pi^1_a\pi^1_b \ , \end{aligned} \tag{7.9}$$

and for $k = \alpha = 2, 3, 4$

$$(b'') \qquad \frac{d\pi^\alpha_j}{dt} = \Gamma^b_{jc} g^{ac}\pi^1_a\pi^\alpha_b \ . \tag{7.10}$$

The pair of equations (7.9-a') and (7.9-b') combine into the second order **geodesic equation**

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \ , \tag{7.11}$$

while the equation (7.10-b") can be rewritten as

$$\frac{D\pi^\alpha_j}{Dt} = \frac{d\pi^\alpha_j}{dt} - \Gamma^k_{ij}\frac{dx^i}{dt}\pi^\alpha_k = 0 \ , \quad \alpha = 2, 3, 4 \ \ .$$

These last equations are just the equations for parallel transport of the 2,3, and 4 legs of a coframe along the geodesic determined by equation (7.11). The result is that $X^1_{\hat{g}}$ **generates parallel transport of linear frames and coframes along time like geodesics of** Γ_g. If we repeat this discussion for, say $X^2_{\hat{g}}$, then again we obtain parallel transport of linear frames along geodesics, but these geodesics will generally be spacelike.

The four Hamiltonian vector fields $X^i_{\hat{g}}$ associated with the spacetime metric tensor can therefore be used to construct the local Lorentzian coordinate systems carried by a freely-falling observer. Let $p_0 \in M$ and let (e_i) be an orthonormal frame at p_0. By integrating $X^1_{\hat{g}}$ with initial condition $(p_0, e_i) \in LM$ we obtain the time-like geodesic $\gamma(s)$ through p_0 determined by e_1, and a parallelly propogated orthonormal spatial triad $(e_\alpha(s))$, $\alpha = 2, 3, 4$, determined by e_2, e_3 and e_4. At each point along $\gamma(s)$ we can then fill in the spatial coordinate axes locally by integrating the vector fields $X^\alpha_{\hat{g}}$ with initial conditions $(\gamma(s), e_i(s))$. Each of these integrations will produce a spatial geodesic through $\gamma(s)$ and the parallel propogation of a triad of vectors along that geodesic. A local coordinate system determined in this way is referred to as **the local Lorentzian coordinate system carried by a freely-falling observer** [20,21]. Because it takes all four Hamiltonian vector fields $X^i_{\hat{g}}$ to determine a coordinate system in this way, it seems appropriate to refer to $\hat{g}$ as the **generalized Hamiltonian tensor for free inertial observers**.

The existence of such local coordinate systems on spacetime is one aspect of Einstein's original correspondence principle. What we have shown above is that the dynamics of such local inertial observers does not have to be postulated, but rather is derivable from generalized Hamilton equations on LM.

8. The Dirac Equation

We recast the fundamentals of the Kostant-Souriau theory of geometric quantization [6,7], taking now for the symplectic manifold the bundle of linear frames LM of spacetime M with the generalized symplectic form $\beta = d\theta$. We restrict attention to the essentials of the initial **pre-quantization** procedure without concern here for the details of a more complete development of a full theory on LM.

In the naive prequantization program one assigns to each observable $f : T^*M \to \mathbb{R}$ a Hermitian operator

$$f \longrightarrow \mathcal{P}_f = -i\hbar X_f \quad . \tag{8.1}$$

The operator $\mathcal{P}_f$ operates as a linear operator on the set of square integrable functions $\psi : T^*M \to \mathbb{C}$, square integrability being defined with respect to the natural Liouville volume element on T^*M. Although the assignment (8.1) is not the completely correct assignment in the full geometric quantization theory (see [6,7]) it will suffice for our purposes here.

We consider a spacetime $(M, \vec{g})$ which admits a spin structure [22], and ask for the prequantization operator assignments that one can make for the metric tensor Hamiltonian observable $\hat{g}$ on LM. The natural analogue of (8.1) is

$$\hat{g} \longrightarrow \mathcal{P}_{\hat{g}} = -i\hbar \hat{X}_{\hat{g}} = -i\hbar X^i_{\hat{g}} \hat{r}_i \quad , \tag{8.2}$$

with the $X^i_{\hat{g}}$ given in (7.6). We consider three **representations** of this operator $\mathcal{P}_{\hat{g}}$. The first and simpliest representation is as the **scalar operator**

$$\mathcal{P}_{\hat{g}} \longrightarrow \mathcal{P}^2_{\hat{g}} \equiv -\hbar^2 \hat{g}_{ij} X^i_{\hat{g}} \circ X^j_{\hat{g}} \tag{8.3}$$

acting on functions that are invariant on fibers of LM. In this case the operator $\mathcal{P}^2_{\hat{g}}$ is proportional to the d'Alembertian operator

$$\mathcal{P}^2_{\hat{g}}(\Psi) = (-\hbar^2)\nabla^2(\Psi) \quad . \tag{8.4}$$

The eigenvalue equation for this operator is then the Klein Gordon equation

$$(-\hbar^2)\nabla^2\Psi = \mu\Psi \quad . \tag{8.5}$$

A second representation of the operator $\mathcal{P}_{\hat{g}}$, which might be called the **vector representation**, can be defined as follows. Let $\vec{t}$ denote a timelike vector field on spacetime M, and let $\hat{t}$ be the corresponding element in T^1 on LM. We consider the operator

$$\mathcal{P}_{\hat{g}} \longrightarrow -\hat{g}_{ij}\hat{t}^i\mathcal{P}^j_{\hat{g}} = i\hbar\hat{g}_{ij}\hat{t}^i X^j_{\hat{g}} = i\hbar\hat{t}^i B_i \quad , \tag{8.6}$$

where the last equality follows from (7.7). It is easy to show that this operator is $i\hbar$ times the **horizontal lift** of $\vec{t}$ to LM relative to the Levi-Civita connection defined by the metric tensor $\vec{g}$. Since $\vec{t}$ is time-like we can think of the operator defined in (8.6) as a relativitic analogue of the Schrödinger energy operator $i\hbar\frac{d}{dt}$.

Finally we consider the **spinor representation** of the operator $\mathcal{P}_{\hat{g}}$, which we define by

$$\mathcal{P}_{\hat{g}} \longrightarrow -\gamma_i\mathcal{P}^i_{\hat{g}} = i\hbar\gamma_i X^i_{\hat{g}} = i\hbar\gamma^i B_i \quad , \tag{8.7}$$

where the four γ_i is a set of appropriate Dirac matrices. In writing (8.7) we are assuming that the vector fields $X^i_{\hat{g}}$ have been lifted to the spin bundle SP(M) over the orthonormal frame subbundle $O_{\hat{g}}(M)$ of LM; that is, the B_i in (8.7) are the standard horizontal vector fields defined by the Levi-Civita connection on SP(M). It follows that (8.7) is the Dirac operator on SP(M) [22].

Let $\Psi : SP(M) \rightarrow \mathbb{C}^4$ be a Dirac 4-spinor transforming under $SL(2,\mathbb{C})$ transformations on the spin bundle as

$$\Psi(u\cdot a) = \rho(a^{-1})\cdot\Psi(u) \quad , \ \forall\ u \in SP(M)\ ,\ \forall\ a \in SL(2,\mathbb{C}) \quad , \tag{8.8}$$

where ρ denotes the 4-spinor representation of $SL(2,\mathbb{C})$. It is straight forward to show that

$$\gamma_i\mathcal{P}^i_{\hat{g}}(\Psi)(u\cdot a) = \rho(a^{-1})\cdot\gamma_i\mathcal{P}^i_{\hat{g}}(\Psi)(u) \quad . \tag{8.9}$$

Thus the eigenvalue equation

$$-\gamma_i\mathcal{P}^i_{\hat{g}}(\Psi) = \mu\Psi \tag{8.10}$$

for the prequantization operator $\gamma_i\mathcal{P}^i_{\hat{g}}$ is equivariant on SP(M) and is in fact just the Dirac equation

$$i\hbar\gamma^i B_i(\Psi) = \mu\Psi \quad . \tag{8.11}$$

9. Conclusions

In this paper we have developed the fundamentals of the generalized symplectic geometry on the bundle of linear frames LM of an n-dimensional manifold M that follows upon taking the $\mathbb{R}^n$-valued soldering 1-form θ on LM as a generalized symplectic potential. This study was motivated by the following two points:

I. An essential feature of quantum mechanics is the unavoidable interaction of observer and object, and in relativistic physics observers are modeled as points in the bundle of linear frames LM of spacetime M.

II. The Kostant-Souriau theory of geometric quantization takes symplectic geometry on phase space as a starting point for the theory.

It is reasonable to suppose that the fundamental quantum mechanical phenomenon of observer-object interaction ought to be able to be studied well on the **manifold of observers**, namely the frame bundle LM of spacetime M. To then carry through a generalization of the Kostant-Souriau theory one would need a symplectic geometry on LM, and we have developed the fundamentals of the geometry with this long range goal in mind. Indeed, as a preliminary application of the generalized symplectic geometry to quantum theory we showed in Section 8 that the Dirac equation arises in a natural way as an eigenvalue equation for a naive prequantization operator assigned to the spacetime metric tensor Hamiltonian.

The heart of standard symplectic geometry is the assignment $f \longrightarrow X_f$ of a Hamiltonian vector field X_f to each real-valued observable f. This is done via the equation

$$df = -X_f \rfloor\ \omega \quad , \tag{9.1}$$

where ω is the symplectic 2-form. Once the assignments are made for each observable one may then proceed to compute Poisson brackets, integrate the equations of motion (find integral curves!), and in general do symplectic geometry. Everything flows from the basic equation (9.1) which may be considered as a structure equation of symplectic geometry. The development of generalized symplectic geometry on LM presented in this paper has been centered around generalizations of (9.1) to LM when ω is replaced by $\beta = d\theta = d\theta^i r_i$ with $\theta = \theta^i r_i$ the $\mathbb{R}^n$-valued soldering 1-form. The fact that $d\theta$ is $\mathbb{R}^n$-valued necessitated generalizing from $\mathbb{R}$-valued observables to vector-valued observables on LM, and we found in Sections 2,3 and 4 that the algebras of symmetric and anti-symmetric contravariant tensor fields on the base manifold have natural interpretations in terms of symplectic geometry on LM.

For a rank p symmetric contravariant tensor field $\vec{f}$ on M we used the uniquely related $\otimes_s^p\mathbb{R}^n$-valued function $\hat{f} = \hat{f}^{i_1\ldots i_p} r_{i_1} \otimes_s \cdots \otimes_s r_{i_p}$ on LM to assign a set of Hamiltonian vector fields $X_{\hat{f}}^{i_1\ldots i_{p-1}}$ via the generalized structure equation

$$d\hat{f}^{i_1\ldots i_p} r_{i_1} = -p! X_{\hat{f}}^{(i_1\ldots i_{p-1}} \lrcorner \, d\theta^{i_p)} \quad . \tag{9.2}$$

For $p > 1$ the $\otimes_s^{p-1}\mathbb{R}^n$-valued Hamiltonian vector fields $X_{\hat{f}} = X_{\hat{f}}^I r_I$ where $I = i_1 \ldots i_{p-1}$, were determined uniquely only locally on LM using the condition (3.11) that depends explicitly on a choice of coordinates. However, the generalized bracket defined by

$$\{\hat{f}, \hat{g}\}^{i_1\ldots i_{p+q-1}} = p! X_{\hat{f}}^{(i_1\ldots i_{p-1}} \left(\hat{g}^{i_p\ldots i_{p+q-1})}\right) \tag{9.3}$$

for $\hat{f} \in ST^p$, $\hat{g} \in ST^q$, produces an element of ST^{p+q-1} which is independent of the choice of coordinates. The result proved in Section 3 is that the algebra $(ST, \otimes_s)$, where $ST = \sum_{p=1}^{\infty} ST^p$ is the vector space of all $\otimes_s^p\mathbb{R}^n$-valued tensorial functions on LM, becomes a **Poisson algebra** under the Poisson bracket defined in (9.3). In addition the set of all locally defined $\otimes_s^{p-1}\mathbb{R}^n$-valued vector fields $\hat{X}_{\hat{f}}$ forms a **Lie algebra** under the Lie bracket defined by

$$[\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] = [X_{\hat{f}}^{i_1\ldots i_{p-1}}, X_{\hat{g}}^{i_p\ldots i_{p+q-1}}] \otimes r_{i_1} \otimes_s r_{i_2} \otimes_s \cdots \otimes_s r_{i_{p+q-1}} \quad . \tag{9.4}$$

In the case of anti-symmetric contravariant tensor fields on M we found that the corresponding vector space $AT = \sum_{p=1}^{\infty} AT^p$ of $\otimes_a^p\mathbb{R}^n$-valued functions on LM becomes a **Poisson super algebra** under the bracket defined by

$$\{\hat{f}, \hat{g}\}^{i_1\ldots i_{p+q-1}} = p! X_{\hat{f}}^{[i_1\ldots i_{p-1}} \left(\hat{g}^{i_p\ldots i_{p+q-1}]}\right) \tag{9.5}$$

for $\hat{f} \in AT^p$ and $\hat{g} \in AT^q$. The component vector fields $X_{\hat{f}}^{i_1\ldots i_{p-1}}$ of the $\otimes_a^{p-1}\mathbb{R}^n$-valued vector field $\hat{X}_{\hat{f}} = X_{\hat{f}}^I r_I$, $I = i_1 \ldots i_{p-1}$, are determined locally by the equation

$$d\hat{f}^{i_1\ldots i_p} r_{i_1} = -p! X_{\hat{f}}^{[i_1\ldots i_{p-1}} \lrcorner \, d\theta^{i_p]} \quad , \tag{9.6}$$

and the set of all such vector fields forms a **super algebra** under the super bracket defined by

$$[\hat{X}_{\hat{f}}, \hat{X}_{\hat{g}}] = [X_{\hat{f}}^{i_1\ldots i_{p-1}}, X_{\hat{g}}^{i_p\ldots i_{p+q-1}}] \otimes r_{i_1} \otimes_a r_{i_2} \otimes_a \cdots \otimes_a r_{i_{p+q-1}} \quad . \tag{9.7}$$

In both of these special cases the brackets, defined in (9.3) and (9.5), are related to differential invariants discovered by Schouten [14] and studied by Nijenhuis [15]. More specifically, expressing the right hand side of (9.3) in coordinates on the base manifold one finds that the Poisson bracket $\{\hat{f}, \hat{g}\}$ corresponds to the **differential concomitant** of the corresponding symmetric contravariant tensor fields $\vec{f}$ and $\vec{g}$ on M due to Schouten and Nijenhuis. Similarly, the rank $r = p+q-1$ anti-symmetric tensor field on M represented by (9.5) is the Schouten-Nijenhuis differential concomitant of the corresponding tensor fields $\vec{f}$ and $\vec{g}$ on M. The generalized symplectic geometry on LM thus provides a natural and unified treatment of the Schouten-Nijenhuis concomitants.

Bhaskara and Viswanath [11] have studied the **symmetric** and **alternating products** of contravariant tensor fields, introduced by Schouten, from an algebraic point of view. Bloore and Assimakopoulos [23], in a study of the Schouten product for symmetric contravariant tensor fields $S\mathcal{X}(M)$ on a manifold M, discovered a natural **1-cochain** that they then used to define the Schouten concomitant "...in exactly the same way as the Poisson bracket is defined using" the canonical 1-form on T^*M. Bloore and Assimakopoulos remarked that they were forced into studying derivations because $S\mathcal{X}(M)$ "...is not an algebra of functions on a manifold and so we cannot set up homology chains involving vector fields." We point out that by working on LM we have replaced $S\mathcal{X}(M)$ with the algebra (under $\otimes_s$) of **functions** ST on LM, which we feel is the essential ingredient in setting up the Poisson algebra $(ST, \{\ ,\ \})$ on LM. The conclusion to be drawn is that the differential geometry of symmetric (anti-symmetric) contravariant tensor fields on a manifold has a natural formulation on LM as a Poisson algebra (Poisson super algebra). The naturally defined brackets then give the Schouten differential concomitants when reinterpreted on the base manifold.

It should be pointed out that the Schouten concomitant for symmetric contravariant tensor fields has long been known [24] to be related to the standard Poisson bracket on T^*M. If $\vec{f} \in S\mathcal{X}^r(M)$, then $\vec{f}$ defines a homogeneous **polynomial observable** [24]

$$\tilde{f} = f^{i_1 \ldots i_r}(x) p_{i_1} \ldots p_{i_r} \tag{9.8}$$

on T^*M in standard canonical coordinates (x^i, p_j). Let $\tilde{f}$ and $\tilde{g}$ be polynomial observables induced on T^*M as in (9.8) by rank r and s symmetric contravariant tensor fields $\vec{f}$ and $\vec{g}$, respectively, on M. Then the Poisson bracket $\{\tilde{f}, \tilde{g}\}$ defined with respect to the canonical symplectic 2-form on T^*M gives the Schouten concomitant of $\vec{f}$ and $\vec{g}$ when re-expressed on M. Note, however, that there is no possibility of obtaining the Schouten concomitant for anti-symmetric tensor fields on T^*M in this way since the right hand side of (9.8) vanishes identically if $f^{i_1 \cdots i_r}(x)$ is anti-symmetric. On the other hand the anti-symmetric case was handled quite satisfactorily in terms of the generalized symplectic geometry on

LM. Generalized symplectic geometry on the frame bundle of a manifold thus unifies and clarifies the many different approaches to the differential concomitants of Schouten.

The homogeneous polynomial observables mentioned above are special cases of the **polynomial observables** that one may define [24,25] on T^*M. We found in Section 6 that the **locally defined allowable observables** on LM for $\otimes^p_s\mathbb{R}^n$-valued functions are polynomials in the generalized momentum coordinates π^i_j with coefficients in the set of functions on LM that are invariant on fibers. The **locally defined allowable observables** on LM for $\otimes^p_a\mathbb{R}^n$-valued functions are exterior products of the π^i_j with coefficients also in the set of functions that are invariant on fibers in LM. Knowledge of these locally defined allowable observables would be important, for example, in setting up canonical commutation relations for a generalized canonical quantization scheme on LM. Although we will not go into the details here, we point out that the natural **canonically conjugate** variables on $(LM, d\theta)$ that generalize the q^i and p_j on T^*M are the $\mathbb{R}^n$-valued coordinate functions

$$\hat{x}^i = x^i r_i = x^i \delta^j_i r_j \quad , \quad \text{(no sum on i)} \quad , \tag{9.9}$$

and the $\mathbb{R}^n$-valued conjugate momentum coordinates

$$\hat{\pi}_i = \pi^j_i r_j \quad . \tag{9.10}$$

Observe that the momentum coordinate $\hat{\pi}_i$, by (9.10) and (2.21), corresponds to the locally defined vector field $\frac{\partial}{\partial x^i}$ on M. On the other hand the canonical coordinate $\hat{x}^i$, by (9.9) and (2.21), **does not** correspond to a vector field on M, but rather corresponds to a locally defined allowable observable in LHF^1.

The generalized Poisson brackets, calculated using $X_{\hat{x}^i} = -\frac{\partial}{\partial \pi^i_i}$ and $X_{\hat{\pi}_i} = \frac{\partial}{\partial x^i}$, are

$$\begin{aligned} \{\hat{x}^i, \hat{x}^j\} &= 0 \quad , \\ \{\hat{\pi}_i, \hat{\pi}_j\} &= 0 \quad , \\ \{\hat{\pi}_j, \hat{x}^i\} &= \delta^i_j r_i \quad , \quad \text{(no sum on i)} \quad . \end{aligned} \tag{9.11}$$

These are the commutation relations that could serve as a starting point for a generalized canonical quantization scheme on LM.

Finally, we mention the obvious fact that the discussions in this paper have just scratched the surface on the complete theory of generalized symplectic geometry on the frame bundle of a manifold. It is hoped that future developments of the subject will validate the conjecture that a deeper understanding of the quantum theory can be obtained by studying the natural Hamiltonian geometry of the manifold of observers.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor R.O. Fulp for many helpful discussions during the preparation of this manuscript.

References

[1] Sternberg, S. (1964). *Lectures on differential geometry.* Prentice-Hall, Englewood Cliffs, New Jersey.

[2] Hermann, R. (1968). *Differential geometry and the calculus of variations.* Academic Press, New York.

[3] Arnol'd, V. I. (1978). *Mathematical methods of classical mechanics.* Springer, New York.

[4] Abraham, R. and Marsden, J. E. (1978). *Foundations of mechanics.* Benjamin Cummings, Reading, Massachuusetts.

[5] Guillemin, V. and Sternberg, S. (1977). *Geometric asymptotics.* A.M.S., Providence, R.I.

[6] Kostant, B. (1970). Quantization and unitary representations. *Lecture notes in mathematics.* Vol. 170, pp. 87-208, Springer, New York.

[7] Souriau, J. -M. (1970). *Structures des systèmes dynamiques.* Dunod, Paris.

[8] Tulczyjew, W.M. (1974). Poisson brackets and canonical manifolds, Bull. Acad. Poln. Sci. Ser. Sci. Math. Astronom. Phys. XXII No. 9, pp. 931-935.

[9] Lichnerowicz, A. (1977). Les Variétiés de Poisson et leurs algèbres de Lie associées, J. Differential Geometry **12**, pp. 253-300.

[10] Hermann, R. (1977). *Toda lattices, cosymplectic manifolds, Backlund transformations and kinks.* Interdisciplinary Mathematics Vol. XV, Math. Sci. Press, Brookline.

[11] Bhaskara, K.H. and Viswanath, K. (1988). Poisson algebras and Poisson manifolds. *Pitman Research Notes in Mathematics.* Vol. 174, Longman Scientific and Technical, Essex.

[12] Guillemin, V. and Sternberg, S. (1980). The moment map and collective motion, Ann. Physics **127**, pp. 220-253.

[13] Guillemin, V. and Sternberg, S. (1984). *Symplectic techniques in physics.* Cambridge University Press, Cambridge..

[14] Schouten, J.A. (1940). Uber differentialkomitanten zweier kontravarianter grossen, Proc. Kon. Ned. Akad. Wet. Amsterdam, **43**, pp. 449-452.

[15] Nijenhuis, A. (1955). Jacobi-type identities for bilinear differential concomitants of certain tensor fields, Indag. Math. **17**, pp. 390-403.

[16] Kobayashi, S. and Nomizu, K. (1963). *Foundations of differential geometry.* Vol. 1. Interscience, New York.

[17] Zumino, B. (1975). Supersymmetry, in *Gauge theories and modern field theory* , edited by R. Arnowitt and P. Nath, MIT Press, Cambridge, Mass.

[18] Corwin, L., Ne'eman, Y., and Sternberg, S. (1975). Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry), Reviews of Modern Physics, **47**, pp. 573-603.

[19] Sommers, P. (1973). On Killing tensors and constants of motion, J. Math. Phys. **14**, p. 787.

[20] Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1970). *Gravitation.* Freeman, San Francisco.

[21] Synge, J. L. (1971). *Relativity: the general theory.* North-Holland, Amsterdam.

[22] Bleecker, D. (1981). *Gauge theory and variational principles*, Addison-Wesley, Reading, Mass.

[23] Bloore, F. J. and Assimakopoulos, M. (1979). A natural one-form for the Schouten concomitant, Internat. J. Theoret. Phys., **18**, pp. 233-238.

[24] Woodhouse, N. (1980). *Geometric quantization.* Clarendon Press, Oxford.

[25] Kostant, B. (1974). In *Géométrie symplectique et physique mathématique* (editor J. -M. Souriau). CNRS, Paris.