The n-symplectic Algebra of Observables

in

Covariant Lagrangian Field Theory‡

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Abstract

n-symplectic geometry on the adapted frame bundle $\lambda : L_\pi E \to E$ of an $n = (m + k)$-dimensional fiber bundle $\pi : E \to M$ is used to set up an algebra of observables for covariant Lagrangian field theories. Using the principle bundle $\rho : L_\pi E \to J^1\pi$ we lift a Lagrangian $\mathcal{L} : J^1\pi \to \mathbf{R}$ to a Lagrangian $L := \rho^*(\mathcal{L}) : L_\pi E \to \mathbf{R}$, and then use L to define a "modified n-symplectic potential" $\hat{\theta}_L$ on $L_\pi E$, the Cartan-Hamilton-Poincaré (CHP) $\mathbf{R}^n$-valued 1-form. If the lifted Lagrangian is non-zero then $(L_\pi E, d\hat{\theta}_L)$ is an n-symplectic manifold. To characterize the observables we define a lifted Legendre transformation ϕ_L from $L_\pi E$ into LE. The image $Q_L := \phi_L(L_\pi E)$ is a submanifold of LE, and $(Q_L, d(\hat{\theta}|_{Q_L}))$ is shown to be an n-symplectic manifold. We prove the theorem that $\hat{\theta}_L = \phi_L^*(\theta|_{Q_L})$, and pull back the reduced canonical n-symplectic geometry on Q_L to $L_\pi E$ to define the algebras of observables on the n-symplectic manifold $(L_\pi E, d\hat{\theta}_L)$. To find the reduced n-symplectic algebra on Q_L we set up the equations of n-symplectic reduction, and apply the general theory to the model of a k-tuple of massless scalar fields on Minkowski spacetime. The formalism set forth in this paper lays the ground work for a geometric quantization theory of fields.

I Introduction

In order to set up a quantization scheme for Lagrangian field theories modeled on the Kostant-Souriau theory of geometric quantization [1, 2] one needs to find an analogue of the algebra of observables $C^\infty(M, \mathbf{R})$ under Poisson bracket and the isomorphic algebra of connection preserving vector fields on a line bundle $L^\times \to M$ over a symplectic manifold (M, ω). In this paper we construct an algebra of observables for covariant Lagrangian field theories using the n-symplectic theory [3, 4, 5, 6, 7, 8, 9] as the engine for the construction. In a companion paper [10] the algebra constructed here is used as the basis for setting up a Kostant-Souriau geometric quantization scheme for covariant Lagrangian field theories. For other geometric approaches to quantization of fields see the work of Kanatchikov [11] who bases his work on the polysymplectic geometry of Günther [12], and also Hrabak [13], whose work is based on the multisymplectic geometry of Gotay et al. [14].

Let $\mathcal{L} : J^1\pi \to \mathbf{R}$ be a field Lagrangian for a section of an $n = m + k$ dimensional fiber bundle $\pi : E \to M$ over the m dimensional manifold M. To use the n-symplectic theory to construct an algebra of observables we lift the Lagrangian on $J^1\pi$ to the bundle of adapted linear frames $L_\pi E$, the subbundle of LE that arises [15, 8, 9] due to the fiber structure of $E \to M$. (Throughout the paper we use the index range convention $i, j = 1, 2, \ldots, m$, $A, B = m+1, m+2, \ldots, m+k$, and $\alpha, \beta = 1, 2, \ldots, m+k$.) A point in $L_\pi E$ is a triple (e, e_i, e_A) where $e \in E$ and (e_i, e_A) is a linear frame for the tangent space to E at e in which the last k vectors (e_A) are vertical on $\pi : E \to M$. The lifting of $\mathcal{L} : J^1\pi \to \mathbf{R}$ to $L_\pi E$ is natural since $L_\pi E$ is known [9] to be an $H = GL(m) \times GL(k)$ principal fiber bundle over $J^1\pi$. If $\rho : L_\pi E \to J^1\pi$ then we put $L := \rho^*(\mathcal{L})$. Using a lifted Legendre transformation we construct the Cartan-Hamilton-Poincaré (CHP) 1-forms $\hat\theta_L$ first introduced in reference [9], and prove the theorem that $(L_\pi E, d\hat\theta_L)$ is an n-symplectic manifold provided the Lagrangian is non-zero. The <u>observables</u> of the theory are then the $\otimes^p \mathbf{R}^n$-valued functions $\hat f$ on $L_\pi E$ that satisfy the n-symplectic structure equation

$$d\hat f = -\frac{1}{p!} X_{\hat f} \,\lrcorner\; d\hat\theta_L \tag{I.1}$$

for some $\otimes^{p-1} \mathbf{R}^n$-valued vector field $\hat X_{\hat f}$. (Here $\lrcorner$ denotes a tensor product in the range and interior product in the domain.) We will show that the set of allowable observables carries a natural graded Poisson algebra structure, and that the set of all corresponding vector-valued Hamiltonian vector fields $\hat X_{\hat f}$ has a natural Lie algebra structure as well.

The plan of the paper is the following. In section II we develop the algebraic structure that is defined by an n-symplectic structure on an N-dimensional manifold P. Such a structure is defined by an $\mathbf{R}^n$-valued 2-form $\hat\omega$ that is both closed and non-degenerate. We will refer to $(P, \hat\omega)$ as an <u>n-symplectic manifold</u>. To illustrate the theory we will carry along the canonical example of the bundle of linear frames $P = LE$ [3, 4, 5, 6] and its canonically defined n-symplectic structure $\hat\omega = d\hat\theta$, where $\hat\theta$ is the $\mathbf{R}^n$-valued soldering 1-form. In the general case we show that $\hat\omega$ defines a Poisson algebra SHF of $\otimes^p_{(\text{sym})} \mathbf{R}^n$-valued functions on P, the observables of the theory, together with a Lie algebra of vector-valued Hamiltonian vector fields. When $P = LE$ and

$\hat{\omega} = d\hat{\theta}$ the observables are symmetric polynomials in the $\mathbf{R}^n$-valued momenta $\hat{\pi}_\alpha$ with coefficients that are constant on the fibers of $\lambda : LE \to E$. The homogeneous polynomial observables in this case correspond uniquely to symmmetric contravariant tensor fields on E, and the Poisson bracket of two such observables on $L_\pi E$ corresponds [6] to the Schouten-Nijenhuis bracket [16, 17] of the corresponding tensor fields on E. There is a corresponding development for $\otimes^p_{(\text{skew})}\mathbf{R}^n$-valued functions on P, which turns out to be a graded Poisson algebra. The direct sum of these two algebras is then a graded Poisson algebra.

In section III we present the relevant details of the canonical n-symplectic geometry on LE, and the reduced subbundle of adapted linear frames $L_\pi E$. In section IV we recall the bundle structure $\rho : L_\pi E \to J^1\pi$ and the definition [9] of the modified soldering 1-forms on $L_\pi E$, which we refer to as the Cartan-Hamilton-Poincaré (CHP) $\mathbf{R}^n$-value 1-form. The CHP 1-form is defined herein as the pull-back, under a lifted Legendre transformation, of the canonical $\mathbf{R}^n$-valued soldering 1-form on LE to $L_\pi E$. We then prove the theorem that $(L_\pi E, d\hat{\theta}_L)$ is an n-symplectic manifold provided L is non-zero. The algebra of observables defined by a Lagrangian is then the graded Poisson algebra defined by this n-symplectic structure.

In order to find the observables defined by a specific Lagrangian we consider in section V the image $Q_L \subset LE$ of $L_\pi E$ under the lifted Legendre transformation. To characterise the n-symplectic observables on $(Q_L, d\hat{\theta})$ we carry out a reduction of the canonical n-symplectic geometry on LE to Q_L. Our method leads to a system of PDE's that characterize those n-symplectic observables on $(LE, d\hat{\theta})$ that restrict to observables on $(Q_L, d(\hat{\theta}|_{Q_L}))$. In section VI we apply the theory to the massless n-tuple of scalar fields on Minkowski spacetime. Section VII contains a brief summary of our results and some ideas for future work.

II n-symplectic geometry

Let P be an N-dimensional manifold, and let $(\hat{r}_\alpha)$ denote the standard basis of $\mathbf{R}^n$, with $1 \le n \le N$. We suppose there exists on P an n-**symplectic structure**, namely an $\mathbf{R}^n$-valued 2-form $\hat{\omega} = \omega^\alpha \otimes \hat{r}_\alpha$ that satisfies the following two conditions:

$$(C-1) \qquad d\omega^\alpha = 0 \quad \forall \ \alpha = 1, 2, \ldots, n \tag{II.2}$$

$$(C-2) \qquad X \lrcorner \ \hat{\omega} = 0 \quad \Leftrightarrow \quad X = 0 \tag{II.3}$$

Definition II.1 *The pair $(P, \hat{\omega})$ is an n-symplectic manifold.*

Remark: In references [3, 4, 5, 6, 7, 8, 9, 15] the term n-symplectic structure refers to the two-form that is the exterior derivative of the $\mathbf{R}^n$-valued soldering 1-form on frame bundles or subbundles of frame bundles. Günther [12] was perhaps the first to consider a manifold with a non-degenerate $\mathbf{R}^n$-valued 2-form, and he used the terms *polysymplectic structure* and *polysymplectic manifold* for the non-dengenerate 2-form and manifold, respectively. In addition, when one adds a few extra conditions to conditions C-1 and C-2 one arrives at a *k-symplectic manifold*. Specifically, if P is required to support an np-dimensional distribution V

such that

$$(C-3) \qquad N = p(n+1)$$

$$(C-4) \qquad \hat{\omega}|_{V \times V} = 0$$

then P is a *k-symplectic manifold* as defined by both de Leon, Salgado, et al. [19] and also by Awane [18]. To make this identification one needs to make the notational changes $n \longrightarrow k$ and $p \longrightarrow n$ in the above discussion. Thus all k-symplectic manifolds are n-symplectic, but not converserly. The canonical frame bundle example $(LE, d\hat{\theta})$ of an n-symplectic manifold introduced in the next paragraph is also a *k-symplectic* manifold. On the other hand the important example of the adapted frame bundle $L_\pi E$ that is central to this paper is n-symplectic, but not k-symplectic. The problem is that the k-symplectic dimensional requirement $N = p(n+1)$ cannot be satisifed on $L_\pi E$.

We will continue to use the name n-symplectic geometry for the structure in definition II.1 in order to emphasis the geometrical and algebraic developments that our approach provides.

Remark: In this section we will carry along the canonical example $P = LE$ where LE is the (n^2+n)-dimensional bundle of linear frames of the n-dimensional manifold E. The bundle of frames LE supports a canonically defined n-symplectic form $\hat{\omega} = d\hat{\theta}$ where $\hat{\theta}$ is the $\mathbf{R}^n$-valued soldering 1-form, and is defined as follows. If X is a tangent vector to LE at $u = (e, e_\alpha)$ then

$$\hat{\theta}_u(X) = e^\alpha(\lambda_*(X))\hat{r}_\alpha \tag{II.4}$$

where (e^α) denotes the coframe dual to the frame (e_α). The soldering form is evidently the frame bundle counterpart of the canonical 1-form θ on T^*E. It has been shown [4] that much of the canonical symplectic geometry on T^*E can be derived from the n-symplectic geometry on LE.

II.1 Canonical coordinates

Awane [18] has proved a generalized Darboux theorem for k-symplectic geometry. Thus in the neighborhood of each point $u \in P$ one can find canonical (or Darboux) coordinates (π_a^α, z^b), $\alpha, \beta = 1, 2, \ldots k$ and $a, b = 1, 2, \ldots n$. With respect to such canonical coordinates $\hat{\omega}$ takes the form

$$\hat{\omega} = (d\pi_a^\alpha \wedge dz^a) \otimes \hat{r}_\alpha \tag{II.5}$$

Hence we have the following locally defined equations:

$$d\pi_a^\alpha = -\frac{\partial}{\partial z^a} \lrcorner \; \omega^\alpha \;, \qquad dz^a = \frac{\partial}{\partial \pi_a^\alpha} \lrcorner \; \omega^\alpha \;, \qquad (\not\Sigma_\alpha \,) \tag{II.6}$$

Remark: The approach used here to characterize algebras of observables requires the existence of such canonical coordinates. From the results in [3] we know that not all functions are *allowable n-symplectic*

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observables, even in the canonical case of frame bundles. Thus, for example, whether or not there exist pairs $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$, $p = 1, 2, \ldots$ that satisfy equation (II.9) below for a general n-symplectic manifold is an existence question, and must be demonstrated for each n-symplectic manifold. The formulas (II.6) will provide local examples of rank 1 solutions of the n-symplectic structure equations (II.9) when either the geometry is specialized to $k = n$-symplectic geometry where a Darboux theorem holds, or when canonical coordinates are simply known to exist. Fortunately in the cases we will consider later in this paper, and in particular on the adapted frame bundle $L_\pi E$, canonical coordinates are known to exist.

Example: On the bundle of linear frames $\lambda : LE \to E$ one can introduce canonical coordinates in the following way. Let (z^α) be a local chart on $U \subset E$. Then on $\tilde{U} = \lambda^{-1}(U)$ define coordinate functions $(\pi_\beta^\alpha, \tilde{z}^\mu)$ by

$$
\begin{aligned}
\pi_\beta^\alpha(u) &:= e^\alpha \left(\left. \frac{\partial}{\partial z^\beta} \right|_{\lambda(u)} \right) \quad \forall\, u = (e, e_\alpha) \in \tilde{U} \\
\tilde{z}^\alpha(u) &:= z^\alpha(\lambda(u)) \quad \forall\, u \in \tilde{U}
\end{aligned}
$$

(II.7)

Following standard notational conventions we will drop the "over tilde" on the lifted coordinates $\tilde{z}^\alpha$ and write simply z^α for both sets of coordinates. With respect to such a coordinate system on LE the soldering 1-form $\hat{\theta}$ has the local coordinate expression

$$\hat{\theta} = (\pi_\beta^\alpha dz^\beta) \otimes \hat{r}_\alpha$$

(II.8)

The n-symplectic 2-form $d\hat{\theta}$ clearly has the canonical form (II.5) in such a coordinate system.

II.2   The Symmetric Poisson Algebra Defined by $\hat{\omega}$

Throughout this section we let $(P, \hat{\omega})$ be an n-symplectic manifold as defined above. It is convenient to introduce the multi-index notation

$$\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{n-\mu}} = \hat{r}_{\alpha_1} \otimes_s \hat{r}_{\alpha_2} \otimes_s \cdots \otimes_s \hat{r}_{\alpha_{n-\mu}} \;, \qquad 0 \leq \mu \leq n-1$$

In addition round brackets around indices $(\alpha\beta\gamma)$ denotes symmetrization over the enclosed indices.

Definition II.2 *For each $p \geq 1$ let SHF^p denote the set of all $(\otimes_s)^p \mathbf{R}^n$-valued functions $\hat{f} = (\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}) = (\hat{f}^{(\alpha_1\alpha_2\ldots\alpha_p)})$ on P that satisfy the equations*

$$d\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p} = -\frac{1}{p!} X_{\hat{f}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}} \,\lrcorner\, \omega^{\alpha_p)}$$

(II.9)

for some set of vector fields $(X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$. We then set

$$SHF = \oplus_{p\geq 1} SHF^p$$

(II.10)

$\hat{f} \in SHF^p$ *will be referred to as a* <u>*symmetric Hamiltonian function of rank p*</u>.

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Example: The locally defined functions $\hat{f}$ that satisfy (II.9) for the canonical n-symplectic manifold $(LE, d\hat{\theta})$ were given in reference [3]. In particular, contrary to the situation in symplectic geometry, not all $(\otimes_s)^p\mathbf{R}^n$-valued functions on LE are compatible with equation (II.9). The $p = 1, 2$ cases will clarify the structure. Let $ST^p(LE)$ denote the vector space of symmetric $(\otimes_s)^p\mathbf{R}^n$-valued GL(n)-tensorial functions on LE that correspond uniquely to symmetric rank p contravariant tensor fields on E. Similarly let $C^\infty(E, (\otimes_s)^p\mathbf{R}^n)$ denote the set of smooth $(\otimes_s)^p\mathbf{R}^n$-valued functions on LE that are constant on fibers of LE. Then

$$
\begin{aligned}
SHF^1 &= T^1(LE) + C^\infty(E, \mathbf{R}^n) & \text{(II.11)} \\
SHF^2 &= ST^2(LE) + T^1(LE) \otimes_s C^\infty(E, \mathbf{R}^n) + C^\infty(E, \mathbf{R}^n \otimes_s \mathbf{R}^n) & \text{(II.12)}
\end{aligned}
$$

For example, if $\hat{f} = (\hat{f}^\alpha) \in SHF^1$ and $\hat{f} = (\hat{f}^{\alpha\beta}) \in SHF^2$, then in canonical coordinates $(\pi^\alpha_\beta, z^\gamma)$ the functions $\hat{f}^\alpha$ and $\hat{f}^{\alpha\beta}$ have the general forms

$$
\hat{f}^\alpha = A^a \pi^\alpha_a + B^\alpha , \qquad \hat{f}^{\alpha\beta} = A^{\mu\nu} \pi^\alpha_\mu \pi^\beta_\nu + B^{\mu(\alpha} \pi^{\beta)}_\mu + C^{\alpha\beta} \tag{II.13}
$$

where A^a, B^α, $A^{\mu\nu} = A^{(\mu\nu)}$, $B^{\mu\nu}$ and $C^{\mu\nu} = C^{(\mu\nu)}$ are all constant on the fibers of $\lambda : LE \to E$ and hence are pull-ups of functions defined on E.

Remark: The analogous results for the n-symplectic form given in (II.5) above are straight forward to work out in canonical coordinates. For the $p = 1$ and $p = 2$ symmetric cases, one finds:

$$
\hat{f}^\alpha = \mathcal{A}^a \pi^\alpha_a + \mathcal{B}^\alpha , \qquad \hat{f}^{\alpha\beta} = \mathcal{A}^{ab} \pi^\alpha_a \pi^\beta_b + \mathcal{B}^{a(\alpha} \pi^{\beta)}_a + \mathcal{C}^{\alpha\beta} \tag{II.14}
$$

where now all coefficients are functions of the coordinates z^a.

Remark: Although $\hat{\omega}$ is non-degenerate in the sense given in equation (II.3) above, because of the symmetrization on the right-hand-side in (II.9) the relationship between $\hat{f}$ and $(X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$ is not unique unless $p = 1$. Given a pair $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$ that satisfies (II.9) one can always add to $X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ vector fields $Y^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ that satisfy the kernel equation

$$
Y^{(\alpha_1\alpha_2\ldots\alpha_{p-1}} \lrcorner \; \hat{\omega}^{\alpha_p)} = 0 \tag{II.15}
$$

to obtain a new pair $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, \bar{X}_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$ that also satisfies (II.9), where

$$
\bar{X}_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} = X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} + Y^{\alpha_1\alpha_2\ldots\alpha_{p-1}}
$$

Hence we associate with $\hat{f} \in SHF^p$ an equivalence class of $(\otimes_s)^{p-1}\mathbf{R}^k$-valued vector fields, which we denote by $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} \hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}]\!]$. We will see below that even though we obtain equivalence classes of Hamiltonian vector fields rather than vector fields, the geometry still carries natural algebraic structures.

Definition II.3 *For each $p \geq 1$ let SHV^p denote the vector space of all equivalence classes of $(\otimes_s)^{p-1}\mathbf{R}^k$-valued vector fields $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2...\alpha_{p-1}}]\!]$ on P that satisfy the equations (II.9) for some $\hat{f} = \hat{f}^{\alpha_1\alpha_2...\alpha_p}\hat{r}_{\alpha_1\alpha_2...\alpha_p} \in SHF^p$. We then set*

$$SHV = \oplus_{p \geq 1} SHV^p \tag{II.16}$$

$[\![\hat{X}_{\hat{f}}]\!]$ will be referred to as the <u>generalized rank p Hamiltonian vector field defined by $\hat{f}$</u>.

Example: The Hamiltonian vector field $\hat{X}_{\hat{f}}$ for the rank 1 element in (II.13) is unique, and has the form

$$\hat{X}_{\hat{f}} = A^\alpha \frac{\partial}{\partial z^\alpha} - \left(\frac{\partial A^\beta}{\partial z^\gamma}\pi_\beta^\alpha + \frac{\partial B^\alpha}{\partial z^\gamma}\right)\frac{\partial}{\partial \pi_\gamma^\alpha} \tag{II.17}$$

The equivalence class of $\mathbf{R}^n$-valued Hamiltonian vector fields corresponding to the rank 2 element in (II.13) on LE has representatives of the form

$$\hat{X}_{\hat{f}}^\alpha = (A^{\mu\nu}\pi_\mu^\alpha + B^{\nu\alpha})\frac{\partial}{\partial z^\nu} - \frac{1}{2}\left(\frac{\partial A^{\mu\beta}}{\partial z^\gamma}\pi_\mu^\alpha\pi_\beta^\nu + \frac{\partial B^{\mu(\alpha}}{\partial z^\gamma}\pi_\mu^{\nu)} + \frac{\partial C^{\alpha\nu}}{\partial z^\gamma}\right)\frac{\partial}{\partial\pi_\gamma^\nu} + Y_\gamma^{\alpha\nu}\frac{\partial}{\partial\pi_\gamma^\nu} \tag{II.18}$$

where $Y_\gamma^{\alpha\beta}$ are functions subject to the constraint

$$Y_\gamma^{(\alpha\beta)} = 0$$

but are otherwise completely arbitrary. The fact that $Y^\alpha = Y_\nu^{\alpha\mu}\frac{\partial}{\partial\pi_\nu^\mu}$ is purely vertical on $\lambda : LE \to E$ follows from (II.15).

Remark: For the n-symplectic rank 2 symmetric observable given above in (II.14), one can check easily that the local coordinate form of a representative $X_{\hat{f}}^\alpha$ of the equivalence class of Hamiltonian vector fields $[\![\hat{X}_{\hat{f}}]\!]^\alpha$ that satisfies (II.9) has the form

$$X^\alpha = (\mathcal{A}^{ab}\pi_a^\alpha + \mathcal{B}^{b\alpha})\frac{\partial}{\partial z^b} - \frac{1}{2}\left(\frac{\partial\mathcal{A}^{ab}}{\partial z^d}\pi_a^\alpha\pi_b^\sigma + \frac{\partial\mathcal{B}^{a(\alpha}}{\partial z^d}\pi_a^{\sigma)} + \frac{\partial\mathcal{C}^{\alpha\sigma}}{\partial z^d}\right)\frac{\partial}{\partial\pi_d^\sigma} + Y^\alpha \tag{II.19}$$

II.2.1 Poisson Brackets

Definition II.4 *For $p, q \geq 1$ define a map $\{\ ,\ \} : SHF^p \times SHF^q \to SHF^{p+q-1}$ as follows. For $\hat{f} = f^{\alpha_1\alpha_2...\alpha_p}\hat{r}_{\alpha_1\alpha_2...\alpha_p} \in SHF^p$ and $\hat{g} = g^{\beta_1\beta_2...\beta_q}\hat{r}_{\beta_1\beta_2...\beta_q} \in SHF^q$*

$$\{\hat{f}, \hat{g}\}^{\alpha_1\alpha_2...\alpha_{p+q-1}} := p! X_{\hat{f}}^{(\alpha_1\alpha_2...\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}...\alpha_{p+q-1})}\right) \tag{II.20}$$

where $\hat{X}_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}}$ is any set of representatives of the equivalence class $[\![\hat{X}_{\hat{f}}]\!]$.

We need to make certain that $\{\hat{f}, \hat{g}\}$ is well-defined. Suppose we have two representatives $X_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}}$ and $\bar{X}_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}} = X_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}} + Y^{\alpha_1\alpha_2...\alpha_{p-1}}$ of $[\![\hat{X}_{\hat{f}}]\!]$. Then it follows easily from (II.15) that

$$\bar{X}_{\hat{f}}^{(\alpha_1\alpha_2...\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}...\alpha_{p+q-1})}\right) = \hat{X}_{\hat{f}}^{(\alpha_1\alpha_2...\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}...\alpha_{p+q-1})}\right)$$

Hence the bracket is independent of choice of representatives. That $\{\hat{f}, \hat{g}\}$ actually is in SHF^{p+q-1} will follow from corollary (II.7) below.

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Definition II.5 *Let* $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2...\alpha_{p-1}}]\!]$ *and* $[\![\hat{X}_{\hat{g}}]\!] = [\![X_{\hat{g}}^{\alpha_1\alpha_2...\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2...\alpha_{p-1}}]\!]$ *denote the equivalence classes of vector-valued vector fields determined by* $\hat{f} \in SHF^p$ *and* $\hat{g} \in SHF^q$, *respectively. Define a bracket* $[\![\ , \ ]\!] : SHV^p \times SHV^q \to SHV^{p+q-1}$ *by*

$$[\![[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]\!] \quad = \quad [\![[\hat{X}_{\hat{f}}^{(\alpha_1\alpha_2...\alpha_{p-1}} , \ \hat{X}_{\hat{g}}^{\alpha_p\alpha_{p+1}...\alpha_{p+q-2})}]\hat{r}_{\alpha_1\alpha_2...\alpha_{p+q-2}}]\!] \tag{II.21}$$

where the "inside" bracket on the right-hand side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives. (Notice the symmetrization over all the upper indices in this equation.)

We again need to show that this bracket is well-defined. This is shown in the following lemma, in which we will need the formula

$$L_{X^{(J}}\omega^{\alpha)} = 0 \tag{II.22}$$

which follows easily from (II.9) and the formula $L_X\omega = X \lrcorner \ d\omega + d(X \lrcorner \ \omega)$. In (II.22) J denotes the multiindex $\alpha_1\alpha_2...\alpha_{p-1}$, and X^J denotes a representative of a rank p Hamiltonian vector field satisfying equations (II.9). The next lemma shows that the bracket defined in (II.21) is (i) independent of choice of representatives, and (ii) closes on the set of equivalence classes of vector-valued Hamiltonian vector fields.

Lemma II.6 *Let* $[\![\hat{X}_{\hat{f}}]\!]$ *and* $[\![\hat{X}_{\hat{g}}]\!]$ *denote the equivalence classes of vector-valued vector fields determined by* $\hat{f} \in SHF^p$ *and* $\hat{g} \in SHF^q$, *respectively. Then*

$$[\![[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]\!] = \frac{(p+q-1)!}{p! \ q!}[\![\hat{X}_{\{\hat{f},\hat{g}\}}]\!] \tag{II.23}$$

Proof We introduce the multiindex notation $I = \alpha_1\alpha_2...\alpha_{p-1}$ and $J = \beta_1\beta_2...\beta_{q-1}$, so that we may use the shorthand notation $X_{\hat{f}}^{\alpha_1\alpha_2...\alpha_{p-1}} = X_{\hat{f}}^I$ and $X_{\hat{g}}^{\beta_1\beta_2...\beta_{q-1}} = X_{\hat{g}}^J$. Then using the identity $L_X(Y \lrcorner \ \omega) = X \lrcorner \ (L_Y\omega) + [X,Y] \lrcorner \ \omega$ for any vector fields X, Y and any 2-form ω, we find:

$$\begin{aligned}
[\hat{X}_{\hat{f}}^{(I}, \hat{X}_{\hat{g}}^{J}] \lrcorner \ \omega^{\alpha)} &= L_{\hat{X}_{\hat{f}}^{(I}}\left(\hat{X}_{\hat{g}}^{J} \lrcorner \ \omega^{\alpha)}\right) - \hat{X}_{\hat{f}}^{(I} \lrcorner \ \left(L_{\hat{X}_{\hat{g}}^{J}}\omega^{\alpha)}\right) \\
&= L_{\hat{X}_{\hat{f}}^{(I}}\left(\hat{X}_{\hat{g}}^{J} \lrcorner \ \omega^{\alpha)}\right) \quad \text{(By formula (II.22))} \\
&= \hat{X}_{\hat{f}}^{(I} \lrcorner \ d(\hat{X}_{\hat{g}}^{J} \lrcorner \ \omega^{\alpha)}) + d(\hat{X}_{\hat{f}}^{(I} \lrcorner \ \hat{X}_{\hat{g}}^{J} \lrcorner \ \omega^{\alpha)}) \\
&= d(\hat{X}_{\hat{f}}^{(I} \lrcorner \ \hat{X}_{\hat{g}}^{J} \lrcorner \ \omega^{\alpha)}) \quad \text{(since } d\left(\hat{X}_{\hat{g}}^{(J} \lrcorner \ \omega^{\alpha)}\right) = d^2\hat{g}^{J\alpha} = 0) \\
&= -\frac{1}{q!}d\left(\hat{X}_{\hat{f}}^{(I} \lrcorner \ d\hat{g}^{J\alpha)}\right) \quad \text{(By (II.9)))} \\
&= -\frac{1}{p! \ q!}d(\{\hat{f},\hat{g}\}^{IJ\alpha)} \quad \text{(By (II.9)))}
\end{aligned}$$

Hence we have shown that for arbitrary representatives of $[\![\hat{X}_{\hat{f}}]\!]$ and $[\![\hat{X}_{\hat{g}}]\!]$

$$d(\{\hat{f},\hat{g}\}^{IJ\alpha)} = -(p! \ q!)[\hat{X}_{\hat{f}}^{(I}, \hat{X}_{\hat{g}}^{J}] \lrcorner \ \omega^{\alpha)} \tag{II.24}$$

Comparing this result with (II.9) we see that

$$\frac{p! \ q!}{(p+q-1)!}[\hat{X}_{\hat{f}}^{(I}, \hat{X}_{\hat{g}}^{J)}]\hat{r}_{IJ} \in [\![\hat{X}_{\{\hat{f},\hat{g}\}}]\!] \tag{II.25}$$

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holds for arbitrary representatives. The lemma follows. ∎

Corollary II.7

$$\{\hat{f}, \hat{g}\} \in SHF^{p+q-1}$$

Proof The corollary follows from (II.24). ∎

Theorem II.8 $(SHV, [\![\ , \ ]\!])$ *is a Lie Algebra.*

Proof The bracket defined in (II.21) is clearly anti-symmetric. To check the Jacobi identity we note that we only need check it for arbitrary representatives, and we may use the very definition (II.21) for the calculation. Since the bracket on the right-hand-side in (II.21) is the ordinary Lie bracket for vector fields, we see that the bracket defined in (II.21) also must obey the identiy of Jacobi. ∎

We can now show that SHF is a Poisson algebra under the bracket defined in (II.20).

Theorem II.9 $(SHF, \{ \ , \ \})$ *is a Poisson algebra over the commutative algebra* $(SHF, \otimes_s)$.

Proof The bracket defined in (II.20) is evidently antisymmetric. To check the Jacobi identity one proceeds exactly as in reference [3], which used a generalization of a proof given in reference [20]. The proof of the Jacobi identity is given in the appendix.

Now the symmetrized tensor product $\otimes_s$ makes SHF into a commutative algebra. If we now consider again elements $\hat{f} \in SHF^p$, $\hat{g} \in SHF^q$ and $\hat{h} \in SHF^r$, then by using definition (II.20) one may show that

$$\{\hat{f}, \hat{g} \otimes_s \hat{h}\} = \{\hat{f}, \hat{g}\} \otimes_s \hat{h} + \hat{g} \otimes_s \{\hat{f}, \hat{h}\} \quad . \tag{II.26}$$

Thus the bracket defined in (II.20) acts as a derivation on the commutative algebra. ∎

Example: In the canonical case $P = LE$ the brackets just defined have a well-known interpretation. As mentioned above the homogeneous elements in SHF^p make up the space $ST^p(LE)$, the symmetric rank p GL(n)-tensorial functions that correspond to symmetric rank p contravariant tensor fields on E. Then $ST = \oplus_{p \geq 1} ST^p \subset SHF$, and the bracket $\{ \ , \ \} : ST^p \times ST^q \to ST^{p+q-1}$ has been shown [6] to be the frame bundle version of the Schouten-Nijenhuis bracket [16, 17] of the corresponding symmetric tensor fields on E.

Remark: There is also a Schouten-Nijenhuis bracket for anti-symmetric contravariant tensor fields on E, and as one might expect this bracket also extends to LE. This leads to a graded n-symplectic Poisson algebra of tensor-valued functions on LE [5].

III   The Canonical n-symplectic structure on $L_\pi E$

In this section we present a few additional details about canonical n-symplectic geometry on frame bundles that will be needed later. An adapted frame at $e \in E$ is a frame where the last k basis vectors are vertical. Note that coordinate frames that come from adapted coordinates are adapted frames. The adapted frame bundle of π, denoted $L_\pi E$, consists of all adapted frames for E.

$$L_\pi E = \{(e, \{e_i, e_A\}) : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } d_u \pi(e_A) = 0\}$$

We will use the same notation $\lambda : L_\pi E \to E$ to denote the restriction of the projection from LE to $L_\pi E$.

$L_\pi E$ is a reduced subbundle of LE [15], the frame bundle of E. As such it is a principal fiber bundle over E. Its structure group is G_v, the nonsingular block lower triangular matrices.

$$G_v = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in GL(m), B \in GL(k), C \in \mathbf{R}^{km} \right\}$$

G_v acts on $L_\pi E$ on the right by

$$(e, \{e_i, e_A\}) \cdot \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \{(e, \{e_i A_j^i + e_A C_j^A, e_A B_B^A\})$$

III.1   Coordinates on $L_\pi E$

If $(z^\alpha) = (x^i, y^A)$ are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $\lambda^{-1}(U)$. _Coframe_ or _n-symplectic momentum_ coordinates $(z^\alpha, \pi_j^i, \pi_j^A, \pi_B^A)$ on $\lambda^{-1}(U)$ are defined as follows. Let $u = (e, \{e_i, e_A\})$ denote a general point in $L_\pi E$. Then

$$z^\alpha(u) = z^\alpha(e) , \quad \pi_j^i(u) = e^i(\frac{\partial}{\partial x^j}) , \quad \pi_B^A(u) = e^A(\frac{\partial}{\partial y^B}) , \quad \pi_j^A(u) = e^A(\frac{\partial}{\partial x^j}) \qquad \text{(III.27)}$$

Here (e^i, e^A) is the coframe dual to (e_i, e_A), and as is customary we have retained the same symbols z^α for the induced horizontal coordinates. Note that the remaining coordinate functions $\pi_A^i(u) = e^i(\frac{\partial}{\partial y^A})$ are identically zero on $L_\pi E$.

Frame or _n-symplectic velocity_ coordinates $(z^\alpha, v_j^i, v_j^A, v_B^A)$ on $\lambda^{-1}(U)$ are defined by:

$$z^\alpha(u) = z^\alpha(e) , \quad v_j^i(u) = e_j(x^i) , \quad v_B^A(u) = e_B(y^A) , \quad v_j^A(u) = e_j(y^A) \qquad \text{(III.28)}$$

The v coordinates, viewed together as a block triangular matrix, form the inverse of the π coordinates above. The blocks have the following relations:

$$v_j^i \pi_k^j = \delta_k^i \qquad\qquad v_j^A \pi_k^j + v_B^A \pi_k^B = 0 \qquad\qquad v_B^A \pi_C^B = \delta_C^A$$

Finally we define _Lagrangian_ coordinates, which are constructed from the previous two coordinate systems.

$$z^\alpha(u) = z^\alpha(e) , \quad u_j^i = \pi_j^i , \quad u_B^A = \pi_B^A , \quad u_j^A = v_i^A \pi_j^i = -v_B^A \pi_j^B \qquad \text{(III.29)}$$

The name _Lagrangian_ coordinates refers to the fact, shown in reference [9], that the u_j^A coordinates are pull-ups, under the projection ρ defined in the next setion, of the standard jet coordinates on $J^1\pi$.

Later in the paper we will need the following formulas for the fundamental vertical vector fields $E_\beta^{*\alpha}$ on $L_\pi E$ in Lagrangian coordinates.

$$E_j^{*i} = -u_k^i \frac{\partial}{\partial u_k^j} \qquad E_B^{*A} = -u_C^A \frac{\partial}{\partial u_C^B} \qquad E_A^{*i} = u_k^i v_A^B \frac{\partial}{\partial u_k^B} \qquad \text{(III.30)}$$

IV The Modified n-symplectic Structure Defined by a Lagrangian L

To bring the Lagrangian into the n-symplectic picture, McLean and Norris [9] showed that $L_\pi E$ is a principal $H = GL(m) \times GL(k)$ bundle over the bundle $J^1\pi$ of 1-jets of sections of π. Letting $\rho : L_\pi E \to J^1\pi$ denote the projection, McLean and Norris then defined the CHP 1-forms θ_L^α on $L_\pi E$ as follows. If $\mathcal{L}$ is a Lagrangian on $J^1\pi$, the the _lifted_ Lagrangian is $L = \rho^*(\mathcal{L})$. Define θ_L^α by:

$$\theta_L^i \;\; := \;\; \tau L\theta^i + E_A^{*i}(L)\theta^A \qquad \qquad \text{(IV.31)}$$

$$\theta_L^A \;\; := \;\; \theta^A \qquad \qquad \text{(IV.32)}$$

where $\tau = \tau(m)$ is a positive constant depending only on the dimension m of the base manifold M, and E_A^{*i} denotes the fundamental vertical vector field on $L_\pi E$ (see III.30 above) corresponding to the element E_A^i in the standard basis (E_β^α) of $gl(n)$. The quantities $E_A^{*i}(L)$, referred to as the "covariant canonical momenta" in [9], are _globally defined_ on $L_\pi E$. In local canonical coordinates (z^α, π_ν^μ), these quantities have the local expressions

$$E_A^{*i}(L) = \pi_j^i p_B^j v_A^B \;\;, \;\; p_B^j = \frac{\partial L}{\partial u_j^B} \qquad \qquad \text{(IV.33)}$$

and clearly are the frame components of the "canonical field momenta" $p_B^j = \frac{\partial L}{\partial u_j^B}$. For different values of τ one can obtain the de Donder-Weyl theory [21, 22] and the Caratheodory theory [23, 22] as special cases of the formalism presented in reference [9]. The significance of these CHP 1-forms as regards other geometrical theories was also considered by MacLean and Norris. In [9] it was shown that one may construct the CHP m-form on $J^1\pi$ from the CHP 1-forms on $L_\pi E$. In this regard see also references [5, 8, 15].

It is clear from the definitions (IV.31) and (IV.32) that the CHP 1-forms have the property that

$$X \lrcorner \; \theta_L^\alpha = 0 \;\; \forall \; \alpha = 1, 2, \ldots, n \; \Longleftrightarrow \; d\lambda(X) = 0$$

Because of this property we can think of the CHP 1-forms as "modified soldering 1-forms", although the θ_L^α may not have the same transformation property under right translation as do the θ^α because of the presence of the Lagrangian L. However, by restricting attention to the H bundle $\rho : L_\pi E \to J^1\pi$, we can show that the θ_L^α are tensorial with respect to H transformations.

Lemma IV.1 _For all $h \in H$ the CHP 1-forms θ_L^α satisfy the tensorial transformation law_

$$R_h^*(\theta_L^\alpha) = \left(h^{-1}\right)_\beta^\alpha \theta_L^\beta \qquad \qquad \text{(IV.34)}$$

Proof

The CHP 1-forms given in (IV.31) and (IV.32) can be expressed, using the Lagrangian coordinates defined in (III.29) above, in the form

$$\theta^i \;=\; u^i_j(-\mathcal{H}^j_k dx^k + p^j_B dy^B) \tag{IV.35}$$

$$\theta^A \;=\; u^A_B(-u^A_k dx^k + dy^B) \tag{IV.36}$$

where we have introduced the definitions

$$\mathcal{H}^i_j := p^i_A u^A_j - \tau L \delta^i_j \;, \qquad p^i_A := \frac{\partial L}{\partial u^A_i} \tag{IV.37}$$

Using the additional definitions

$$\left(h^\alpha_\beta\right) = \begin{pmatrix} -\mathcal{H}^k_j & p^k_A \\ -u^E_j & \delta^E_A \end{pmatrix} \;, \qquad \left((\Delta u)^\alpha_\beta\right) = \begin{pmatrix} u^k_j & 0 \\ 0 & u^E_A \end{pmatrix} \tag{IV.38}$$

equations (IV.35) and (IV.36) can be written in the following compact form:

$$\theta^\alpha_L = ((\Delta u)^\alpha_\beta) h^\beta_\gamma dz^\gamma \tag{IV.39}$$

The matrix $((\Delta u)^\alpha_\beta)$ in (IV.39) transforms under the group H of the bundle $\rho : L_\pi E \to J^1\pi$ while the second factor (h^α_β) is H-invariant. In particular, $R^*_h((\Delta u)^\alpha_\mu) = \left(h^{-1}\right)^\alpha_\beta (\Delta u)^\beta_\mu$. The lemma follows. ∎

The lemma shows that the CHP 1-forms do behave like modified soldering 1-forms with respect to the bundle $\rho : L_\pi E \to J^1\pi$. The geometrical significance of the lemma is that the CHP-forms define a set of type 1-1 tensor fields on the jet bundle $J^1\pi$. To see this we recall that the canonical soldering 1-forms θ^α define the type 1-1 *identity tensor field* on E. The construction is as follows. Let $u = (e, e_\beta) \in LE$ be an arbitrary point in the bundle of frames of E. The coframe to (e_β) may be written as $(e^\beta = \pi^\beta_\alpha(u) dz^\alpha)$. Then $\hat\theta = (\theta^\alpha)$ defines the identity type 1-1 tensor field on E as follows:

$$\hat\theta(u) = (\pi^\alpha_\beta dz^\beta)(u) \otimes r_\alpha \longrightarrow \pi^\alpha_\beta(u) dz^\beta \otimes e_\beta = e^\beta \otimes e_\beta = I_e \tag{IV.40}$$

The tensorial transformation property of $\hat\theta$ on LE or any of its subbundles guarantees that the 1-1 tensor field defined in (IV.40) is well-defined. We can use a similar construction to define a 1-1 tensor field on $J^1\pi$ based on the CHP 1-forms.

Let $u = (e, e_\beta) \in L_\pi E$ be an arbitrary point in the bundle of adapted linear frames of E, and let $v = \rho(u) \in J^1\pi$ be the projection to $J^1\pi$. Then we define a 1-1 tensor field T on $J^1\pi$ as follows:

$$\hat\theta_L(u) \longrightarrow T(v) = \theta^i_L(u) \otimes e_i + \tau L(u)\theta^A_L \otimes e_A = ((\Delta u)^i_\beta)(u) h^\beta_\gamma(u) dz^\gamma \otimes e_i + \tau L(u)((\Delta u)^A_\beta)(u) h^\beta_\gamma(u) dz^\gamma \otimes e_A \tag{IV.41}$$

Notice the inclusion of the H-invariant factor $\tau L(u)$ in the definition.

It is easy to show from the definitions that $((\Delta u)^i_\beta)(u) e_i = \delta^i_\beta(\frac{\partial}{\partial x^i} + u^A_i(v)\frac{\partial}{\partial y^A})$ and $((\Delta u)^A_\beta)(u) e_A = \delta^A_\beta \frac{\partial}{\partial y^A}$. Using these results together with the definition (IV.37) in the last equation we can rewrite it as

$$T(v) = (-\mathcal{H}^i_j) dx^k \otimes \frac{\partial}{\partial x^i} - (u^B_k p^j_B u^A_j) dx^k \otimes \frac{\partial}{\partial y^A} + (p^j_A) dy^A \otimes \frac{\partial}{\partial x^j} + (p^j_B u^A_j + \tau L \delta^A_B) dy^B \otimes \frac{\partial}{\partial y^A} \tag{IV.42}$$

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T is well-defined since the right hand side of the definition of $T(v)$ is H-invariant. Hence the CHP 1-forms define a type 1-1 tensor field on $J^1\pi$, whose independent components with respect to any canonical chart on $L_\pi E$ are, after omitting minus signs:

(a) $\qquad (\mathcal{H}^i_j)dx^k \otimes \dfrac{\partial}{\partial x^i}$ $\qquad$ the energy-momentum tensor of the field $\hfill$ (IV.43)

(b) $\qquad (p^j_A)dy^A \otimes \dfrac{\partial}{\partial x^j}$ $\qquad$ the canonical field momentum $\hfill$ (IV.44)

(c) $\qquad (u^B_k p^j_B u^A_j)dx^k \otimes \dfrac{\partial}{\partial y^A}$ $\qquad$ the canonical field momentum summed with field velocities$\hfill$(IV.45)

(d) $\qquad (p^j_B u^A_j + \tau L\delta^A_B)dy^B \otimes \dfrac{\partial}{\partial y^A}$ $\qquad$ a new energy-momentum-type field

These tensor fields will be of importance in the application presented in section VI.

IV.1 The Legendre Transformation

One can define the CHP 1-forms using a frame bundle version of the Legendre transformation. Given a Lagrangian $L : L_\pi E \to \mathbf{R}$ we obtain a mapping $\phi_L : L_\pi E \to LE$ given by

$$\phi_L(u) = \phi_L(e, e_i, e_A) = \left(e, \frac{1}{\tau L(u)}e_i, e_A - \frac{1}{\tau L(u)}E^{*a}_A(L)(u)e_a\right) \qquad \text{(IV.46)}$$

The condition that this mapping end up in LE is that the Lagrangian be **non-zero**, and *for the rest of this paper we will assume this condition*. We will refer to this mapping as the *n-symplectic Legendre transformation*. Our goal is to prove Theorem (IV.6), namely that $\hat\theta_L = \phi^*_L(\hat\theta)$ where $\hat\theta$ is the canonical soldering 1-form on the image of ϕ_L. This will follow easily once we exhibit the manifold structure of Q_L.

Lemma IV.2 *If the lifted Lagrangian is nonzero, then the Legendre transformation (IV.46) is one-one.*

Proof If $\phi_L(u) = \phi_L(\bar u)$ then the two adapted frames must project to the same point in E. Equating vectors in the frame we find $\frac{1}{\tau L(\bar u)}\bar e_i = \frac{1}{\tau L(u)}e_i$ and $\bar e_A - \frac{1}{\tau L(\bar u)}E^{*a}_A(L)(\bar u)\bar e_a = e_A - \frac{1}{\tau L(u)}E^{*a}_A(L)(u)e_a$. Using the first of these relations in the second and rearranging we obtain

$$\bar e_A - e_A = \left(E^{*a}_A(L)(\bar u) - E^{*a}_A(L)(u)\right)\frac{1}{\tau L(u)}e_i$$

Since both $\bar e_A$ and e_A are vertical on E this implies that $\left(E^{*a}_A(L)(\bar u) - E^{*a}_A(L)(u)\right) = 0$. Hence $\bar e_A = e_A$ and $\frac{1}{\tau L(\bar u)}\bar e_i = \frac{1}{\tau L(u)}e_i$. This implies that $\bar u = u \cdot h$ for $h \in H \subset G_V$. But since the Lagrangian L is lifted, it is H invariant and so $L(\bar u) = L(u)$, which implies that $\bar u = u$. $\blacksquare$

To clarify the meaning of the Legendre transformation (IV.46) we introduce a new manifold $\tilde P$ as follows. Let J denote the subgroup of $GL(n)$ consisting of matrices of the form

$$\begin{pmatrix} I & \xi \\ 0 & I \end{pmatrix} \qquad \xi \in \mathbf{R}^{m\times k}$$

Define $\tilde{P}$ by

$$\tilde{P} = L_\pi E \cdot J = \{(e_i, e_A + \xi_A^j e_j) \mid (e_i, e_A) \in L_\pi E, \ \xi \in \mathbf{R}^{m \times k}\} \tag{IV.47}$$

Lemma IV.3 $\tilde{P}$ *is a open dense submanifold of the bundle of frames LE of E.*

Proof Private communication from Mike McLean. ∎

Lemma IV.4 *There is a canonical diffeomorphism from $\tilde{P}$ to the product manifold $L_\pi E \times \mathbf{R}^{m \times k}$.*

Proof If (e, e_i, e_A) is a point in $L_\pi E$, then we let $\tilde{e}_i = \pi_*(e_i)$. From the structure of $L_\pi E$ it is clear that $(\tilde{e}_i)$ is a linear frame for the tangent space to M at $\pi(e)$. Let $(\tilde{e}^i)$ denote the coframe dual to $(\tilde{e}_i)$. Now suppose $\bar{u} = (\bar{e}_i, \bar{e}_A) = (e_i, e_A + \xi_A^j e_j)$ is an arbitrary point in $\tilde{P}$. Then we have $\pi_*(\bar{e}_i) = \tilde{e}_i$. Using the fact that each e_A is vertical we have $\pi_*(\bar{e}_A) = \xi_A^i \tilde{e}_i$. Hence

$$\xi_A^i = \tilde{e}^i\left(\pi_*(\bar{e}_A)\right) \tag{IV.48}$$

Define a mapping from $\tilde{P}$ to $L_\pi E \times \mathbf{R}^{m \times k}$ by

$$\bar{u} = (\bar{e}_i, \bar{e}_A) \longrightarrow \left((e_i, e_A), \xi_A^i\right) = \left((\bar{e}_i, \bar{e}_A - \tilde{e}^j\left(\pi_*(\bar{e}_A)\right)\bar{e}_j), \tilde{e}^i\left(\pi_*(\bar{e}_A)\right)\right) \tag{IV.49}$$

The mapping (IV.49) is easily shown to be 1-1, and it is clearly smooth. The inverse mapping is the multiplication mapping $\mu : L_\pi E \times \mathbf{R}^{m \times k} \to \tilde{P}$ defined by

$$\mu((e_i, e_A), \xi_B^j) = (e_i, e_A + \xi_A^j e_j) \tag{IV.50}$$

This inverse is evidently smooth. ∎

Remark: The coordinate expression for this mapping is

$$\bar{u} = (\bar{e}_i, \bar{e}_A) \longrightarrow \left((e_i, e_A), \xi_A^i\right) = \left((\bar{e}_i, \bar{e}_B - \bar{\pi}_A^j(\bar{u})\bar{V}_B^A(\bar{u})\bar{e}_j), -\bar{\pi}_A^j(\bar{u})\bar{V}_B^A(\bar{u})\right) \tag{IV.51}$$

where $\bar{V}_B^A(\bar{u}) = v_B^A(u)$ are the components of the matrix inverse of the matrix $\left(\bar{\pi}_B^A(\bar{u})\right) = \left(\pi_B^A(u)\right)$, which must necessarily be non-singular because of the structure of $L_\pi E$. In the following we let $\bar{\rho} : L_\pi E \times \mathbf{R}^{m \times k} \to L_\pi E$ be the natural projection.

Suppose we are given a Lagrangian L on $L_\pi E$. Then it is easy to see that the Legendre transformation (IV.46) can be expressed as the composition $\phi_L = \mu \circ \phi_2 \circ \phi_1$, where μ is the multiplication map defined in (IV.50) above, ϕ_1 is the bundle automorphism

$$\phi_1 : L_\pi E \longrightarrow L_\pi E, \qquad \phi_1(e_i, e_A) = (\frac{1}{\tau L(u)}e_i, e_A) \tag{IV.52}$$

and the mapping ϕ_2 is the global section of $\bar{\rho}$ given by

$$\phi_2(u) = \phi_2(e_i, e_B) = \left((e_i, e_B), -\frac{E_A^{*j}(L)(u)}{\tau L(u)}\right) \tag{IV.53}$$

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The mapping ϕ_1 is 1-1 since the Lagrangian L is invariant under the subgroup H, and $\bar{u} = \phi_1(u) = u \cdot h$ for

$$h = \begin{pmatrix} \tau L(u)I & 0 \\ 0 & I \end{pmatrix}.$$

Q_L is then the image of $L_\pi E$ in $\tilde{P}$ under the C^∞ Legendre transformation. In particular, Q_L is the smooth image under the multiplication map of the global section $\phi_2(L_\pi E)$ of $\bar{\rho} : L_\pi E \times \mathbf{R}^{m \times k} \to L_\pi E$, and hence is a smooth manifold. The inverse of the Legendre transformation is then the composition $\phi_L^{-1} = \phi_1^{-1} \circ \phi_2^{-1} \circ \mu^{-1}$, where ϕ_2^{-1} is the projection $\bar{\rho}$ restricted to $\mu^{-1}(Q_L,)$ and $\phi_1^{-1}(\bar{u}) = \phi_1^{-1}(\bar{e}_i, \bar{e}_A) = (L(\bar{u})\bar{e}_i, \bar{e}_A)$. The inverse is thus also C^∞, and we have:

Lemma IV.5 *If the Lagrangian L is non-zero, then the Legendre transformation $\phi_L : L_\pi E \to Q_L$ is a diffeomorphism.*

Theorem IV.6 *Let L be the pull-up of a non-zero Lagrangian $\mathcal{L}$ on $J^1\pi$, and let ϕ_L denote the Legendre transformation defined above in (IV.46). Then*

$$\hat{\theta}_L = \phi_L^*(\hat{\theta}) \tag{IV.54}$$

Proof A straightforward calculation.

Remark: This theorem has an obvious analogue in symplectic mechanics, where the symplectic form on the velocity phase space TE is, for a regular Lagrangian, the pull back under the Legendre transformation of the canonical 1-form on T^*M. There is also a similar theorem in *multisymplectic geometry* where the CHP m-form on $J^1\pi$ is known [14] to be the pull back of the canonical multisymplectic m-form on $J^{1*}\pi$.

Now Q_L, being a submanifold of LE, supports the restriction $\hat{\theta}|_{Q_L}$ of the $\mathbf{R}^n$-valued soldering 1-form $\hat{\theta}$. It is easy to verify that the closed $\mathbf{R}^n$-valued 2-form $d\hat{\theta}|_{Q_L}$ is also non-degenerate, and hence $(Q_L, d(\hat{\theta}|_{Q_L}))$ is an n-symplectic manifold. Using the fact that Q_L and $L_\pi E$ are diffeomorphic under the Legendre transformation, we obtain the following corollary to Theorem IV.6.

Corollary IV.7 $(L_\pi E, d\hat{\theta}_L)$ *is an n-symplectic manifold.*

Remark: It is also not difficult to show by direct calculuation that $d\hat{\theta}_L$ is non-degenerate.

We would now like to be able to find the n-symplectic observables defined by the n-symplectic structure $d\hat{\theta}_L$ on $L_\pi E$. However, since the new n-symplectic structure is not in canonical form in standard canonical coordinates (z^α, π_ν^μ), it is rather difficult to find the observables. This fact can be clarified as follows.

The local forms given above in (II.13) of the canonical n-symplectic algebras on LE were given in [3], and were found by solving the equations

$$L_{X_{\hat{f}}^{(I}} d\theta^{\alpha)} = 0 \tag{IV.55}$$

for the locally n-symplectic Hamiltonian vector fields. Lawson [15] used the same technique to characterize the reduced algebra on $L_\pi E$ by solving the equations

$$L_{X_{\hat{f}}^{(I}} d(\theta|_{L_\pi E})^{\alpha)} = 0 \tag{IV.56}$$

In both cases the equations were tractable because the n-symplectic forms could be written in canonical coordinates. On the other hand the equations $L_{X_{\hat{f}}^{(I}} d\theta_L^{\alpha)} = 0$ are very complicated and are not easily solved. To get around this problem we will use Corollary (IV.7). If we identify the n-symplectic algebra on the n-symplectic manifold $(Q_L, d(\hat{\theta}|_{Q_L}))$, then we can use the Legendre transformation to transform back to $L_\pi E$. In particular, if $\hat{f}$ and $\hat{X}_{\hat{f}}$ satisfy the equation

$$d\hat{f} = -\hat{X}_{\hat{f}} \lrcorner \ d(\hat{\theta}) \tag{IV.57}$$

on Q_L, then $\phi_L^*(\hat{f})$ and $\phi_{L*}^{-1}(\hat{X}_{\hat{f}})$ satisfy the equation

$$d(\phi_L^*(\hat{f})) = -\phi_{L*}^{-1}(\hat{X}_{\hat{f}}) \lrcorner \ d(\phi_L^*(\hat{\theta})) \tag{IV.58}$$

on $L_\pi E$.

V n-symplectic Reduction

In order to characterize the n-symplectic observables on $(Q_L, d(\hat{\theta}|_{Q_L}))$ we will find a reduction of the n-symplectic geometry of $(\tilde{P}, d\hat{\theta})$ to $(Q_L, d(\hat{\theta}|_{Q_L}))$. Let $i : Q_L \to \tilde{P}$ be the inclusion mapping, and let $\hat{f} : \tilde{P} \to \mathbf{R}^n$ be an $\mathbf{R}^n$-valued observable for the canonical n-symplectic geometry on $(\tilde{P}, d\hat{\theta})$ such that its Hamiltonian vector field $\hat{X}_{\hat{f}}$ is tangent to Q_L at points of Q_L. Then on Q_L we have

$$d(i^*\hat{f}) = -\hat{X}_{\hat{f}} \lrcorner \ d(i^*\hat{\theta}) \tag{V.59}$$

This follows from the fact that $i_* \left(i_*^{-1}(\hat{X}_{\hat{f}}) \right) = \hat{X}_{\hat{f}}$ at points of Q_L when $\hat{X}_{\hat{f}}$ is tangent to Q_L. Let $SHF|_{Q_L}$ be the restriction to Q_L of the subset of SHF on $\tilde{P}$ such that the corresponding Hamiltonian vector fields are tangent to Q_L at points of Q_L. The set $SHF|_{Q_L}$ will be the reduced symmetric n-symplectic algebra defined by the Lagrangian L. To find the reduced algebra we will derive the equations that $\hat{X}_{\hat{f}}$ must satisfy in order to be tangent to Q_L.

Define new coordinates on Q_L using the Legendre transformation to pull back the Lagrangian coordinates $(z^\alpha, u_j^i, u_k^A, u_B^A)$ define in III.29 above. We let

$$\tilde{u}^\alpha = z^\alpha \circ \phi_L^{-1} \ , \quad \tilde{u}_j^i = u_j^i \circ \phi_L^{-1} \ , \quad \tilde{u}_B^A = u_B^A \circ \phi_L^{-1} \ , \quad \tilde{u}_j^A = u_j^A \circ \phi_L^{-1} \tag{V.60}$$

Then the local vector fields

$$\frac{\partial}{\partial \tilde{u}^\alpha} \ , \quad \frac{\partial}{\partial \tilde{u}_j^i} \ , \quad \frac{\partial}{\partial \tilde{u}_j^A} \ , \quad \frac{\partial}{\partial \tilde{u}_B^A} \tag{V.61}$$

form a local basis of the tangent spaces of Q_L. Any vector field that is tangent to Q_L can be expressed locally in terms of this local basis. We first consider a rank 1 Hamiltonian vector field on $\tilde{P}$ given by (see II.17)

$$\hat{X}_{\hat{f}} = f^\alpha \frac{\partial}{\partial \bar{z}^\alpha} - \left(\frac{\partial f^\beta}{\partial \bar{z}^\gamma} \bar{\pi}^\alpha_\beta \right) \frac{\partial}{\partial \bar{\pi}^\alpha_\gamma} \tag{V.62}$$

corresponding to the observable $\hat{f}^\alpha = f^\beta(e)\bar{\pi}^\alpha_\beta$. We suppose that this vector field can be expanded in the basis V.61 and write

$$\hat{X}_{\hat{f}} = X^\alpha \frac{\partial}{\partial \tilde{u}^\alpha} + X^i_j \frac{\partial}{\partial \tilde{u}^i_j} + X^A_B \frac{\partial}{\partial \tilde{u}^A_B} + X^A_j \frac{\partial}{\partial \tilde{u}^A_j} \tag{V.63}$$

Equating these two forms for $\hat{X}_{\hat{f}}$ and using the fact (see IV.39) that $\bar{\pi}^\alpha_\beta \circ \phi_L = (\Delta u)^\alpha_\sigma h^\sigma_\beta$ we find $X^\alpha = f^\alpha$ and

$$-\frac{\partial f^\sigma}{\partial \bar{z}^\beta} \bar{\pi}^\alpha_\sigma = X^\sigma \widetilde{\frac{\partial h^\kappa_\beta}{\partial z^\sigma}} (\Delta \tilde{u})^\alpha_\kappa + X^i_j \delta^\alpha_i \widetilde{h^j_\beta} + X^A_B \delta^\alpha_A \widetilde{h^B_\beta} + X^A_k (\Delta \tilde{u})^\alpha_\kappa \widetilde{\frac{\partial h^\kappa_\beta}{\partial u^A_k}} \tag{V.64}$$

where the "over tildes" indicate that the term is to be evaluated in the new coordinates (V.60). These equations can be solved for the coefficients X^i_j, X^A_B, and X^A_k in terms of the components of $\frac{\partial f^\sigma}{\partial \bar{z}^\beta}$, plus the following <u>constraint</u> equation:

$$-\frac{\partial f^\sigma}{\partial \bar{z}^B} \tilde{h}^b_\sigma = f^\sigma \widetilde{\frac{\partial p^b_B}{\partial z^\sigma}} - \frac{df^b}{d\bar{z}^i} \tilde{p}^i_B - \frac{1}{\tau L} \left(f^\sigma \widetilde{\frac{\partial L}{\partial z^\sigma}} + \frac{df^\sigma}{\bar{z}^k} \tilde{p}^i_A \tilde{h}^A_\sigma \right) \tilde{p}^b_B + \frac{df^\sigma}{d\bar{z}^k} \tilde{h}^A_\sigma \widetilde{\frac{\partial p^b_B}{\partial u^A_k}} \tag{V.65}$$

We next consider a rank 1 Hamiltonian vector field on P given by (see II.17)

$$\hat{X}_{\hat{f}} = \frac{\partial \xi^\alpha}{\partial \bar{z}^\gamma} \frac{\partial}{\partial \bar{\pi}^\alpha_\gamma} \tag{V.66}$$

corresponding to the observable $\hat{f}^\alpha = \xi^\alpha(e)$. Carrying out the same calculuation that we did for $\hat{f}^\alpha = f^\beta(e)\bar{\pi}^\alpha_\beta$ above we find the <u>constraint</u> equation

$$-\tau L \frac{\partial \xi^a}{\partial \bar{z}^D} = \tilde{p}^e_D \left(-\frac{d\xi^a}{d\bar{z}^e} + \tilde{u}^a_b \frac{d\xi^E}{d\bar{z}^k} \tilde{v}^A_E (-\tilde{p}^k_A \delta^b_e + \tilde{p}^b_A \delta^k_e) \right) + \tau L \frac{d\xi^E}{d\bar{z}^k} \tilde{v}^A_E \tilde{u}^a_b \widetilde{\frac{\partial p^b_D}{\partial u^A_k}} \tag{V.67}$$

An n-symplectic observable $\hat{f} = (f^\beta \bar{\pi}^\alpha_\beta + \xi^\alpha)r_\alpha$ on $\tilde{P}$ will be in $SHF|_{Q_L}$ if and only if it satisfies the two constraint equations V.65 and V.67.

VI Application: k-tuple of Massless Scalar Fields on Flat Space-time

As an application of the general theory we will study the rank-1 n-symplectic subalgebra of observables that is defined by the simple model of a k-tuple of massless scalar fields on Minkowski spacetime. Analysis of the more complicated higher rank portions of the algebra will be left for future work. For the k-tuple of massless scalar fields the bundle $\pi : E \to M$ over Minkowski spacetime M is a trivial vector bundle with standard fiber $\mathbf{R}^k$. Such a system has a <u>**lifted**</u> Lagrangian of the form $L = \frac{1}{2}g^{ab}\delta_{AB}u^A_a u^B_b$ where g^{ab} are the contravariant components of the Minkowski metric tensor in arbitrary coordinates on the spacetime manifold and δ_{AB} are

the components of the Euclidean metic tensor for the internal space $\mathbf{R}^k$. Since global inertial coordinates exist on Minkowshi spacetime, we will for the calculations restrict attention to canonical coordinates on LE that are induced by such inertial coordinates on spacetime. Then the components of the metric tensor field will take the constant Minkowski form $(\eta^{ab}) = \text{diag}(-1, 1, 1, 1)$, and the Lagrangian will be

$$L = \frac{1}{2}\eta^{ab}\delta_{AB}u_a^A u_b^B \tag{VI.68}$$

Using this Lagrangian for the k-tuple of scalar fields on Minkowski spacetime we find the standard result

$$p_A^i = \eta^{ij}\delta_{AB}u_j^B \tag{VI.69}$$

We observe that for this Lagrangian equations (V.65) and (V.67) are satisfied by $f^\alpha = C^\alpha = \text{constant}$ and $\xi^\alpha = K^\alpha = \text{constant}$.

Theorem VI.1 *The rank-1 n-symplectic Hamiltonian vector field (II.17) for the n-symplectic manifold $(\tilde{P}, d\hat{\theta})$ will satisfy the reduction equations (V.65) and (V.67) for the Lagrangian of the k-tuple of massless scalar fields on Minkowski spacetime if*

$$f^\alpha = C^\alpha = constant \;, \quad \xi^\alpha = K^\alpha = constant \tag{VI.70}$$

Remark: We recall from equation (II.11) that the rank-1 algebra $HF^1 = T^1(LE) \oplus C^\infty(E, \mathbf{R}^{m+k})$ is the direct sum of the rank-1 tensorial functions $T^1(LE)$ on LE that correspond uniquely to vector fields on E, with the $\mathbf{R}^{m+k}$-valued functions $C^\infty(LE, \mathbf{R}^{m+k})$ that are constant on fibers of LE, i.e. $C^\infty(E, \mathbf{R}^{m+k})$. The above theorem tells us that the subalgebra corresponding to the tensorial functions is a copy of the translation symmetry group $\mathbf{R}^{m+k}$ of the base manifold $E = \mathbf{R}^{m+k}$. Hence we may interpret the tensorial part of the rank-1 subalgebra corresponding to $f^\alpha = C^\alpha = \text{constant}$ as the space of translational Killing vectors for the metrics η on M and δ on the fibers of E. The other part of the algebra, characterized by $\xi^\alpha = K^\alpha = \text{constant}$, can be identified also with $\mathbf{R}^{m+k}$. The part of the rank-1 algebra $SHF^1|_{Q_L}$ on Q_L determined by the theorem is therefore

$$\mathcal{G} = \mathbf{R}^{m+k} \oplus \mathbf{R}^{m+k} \tag{VI.71}$$

There is an alternative interpretation of the tensorial part of the rank-1 subalgebra in terms of "n-symplectic momentum mapping". In [5] it was pointed out that the translation group $\mathbf{R}^{m+k}$ lifts from $E = \mathbf{R}^{m+k}$ to LE to define an n-symplectic momentum mapping, and that for each $(\zeta^\alpha) \in \mathbf{R}^n$ the corresponding "momentum" is $\zeta^\alpha \pi_\alpha^\beta \hat{r}_\beta$. Notice that $\zeta^\alpha \pi_\alpha^\beta \hat{r}_\beta$ is precisely $\hat{\zeta}$, i.e. it is the tensorial function corresponding to the vector field $\zeta^\alpha \frac{\partial}{\partial z^\alpha}$ on E. Hence the tensorial part of the rank-1 subalgebra can also be thought of as the "set of all momenta" that arise from the n-symplectic momentum map defined by the lift of $\mathbf{R}^{m+k}$ to LE.

VI.1 The Algebra of Observables on $L_\pi E$

To obtain the n-symplectic algebra on $L_\pi E$ we must pull-back the observables $\hat{f}$ and use the inverse Legendre transformation to map the corresponding Hamiltonian vector fields $\hat{X}_{\hat{f}}$ to $L_\pi E$. Letting C^α and K^α denote

constants, we have found rank 1 observables on $L_\pi E$ of the form

$$\hat{F} := \phi_L^*(C^\alpha \bar{\pi}_\alpha^\beta + K^\alpha)\ r_\alpha = \left(C^\alpha \phi_L^*(\bar{\pi}_\alpha^\beta) + K^\alpha\right)\ r_\alpha$$

From (IV.39) and (IV.54) one can infer that $\phi_L^*(\bar{\pi}_\beta^\alpha) = (\Delta u)_\sigma^\alpha h_\beta^\sigma$. Hence the rank 1 observables on $L_\pi E$ have the form

$$\hat{F} = \left((C^\beta (\Delta u)_\sigma^\alpha h_\beta^\sigma) + K^\alpha\right)\ r_\alpha \tag{VI.72}$$

Mapping the Hamitonian vector field $\hat{X}_{\hat{f}} = C^\alpha \frac{\partial}{\partial \bar{z}^\alpha}$ corresponding to $(C^\alpha \bar{\pi}_\alpha^\beta + K^\alpha)\ r_\alpha$ to $L_\pi E$ using $(\phi_L^{-1})_*$ one finds in this special case the simple result

$$(\phi_L^{-1})_*(C^\alpha \frac{\partial}{\partial \bar{z}^\alpha}) = C^\alpha \frac{\partial}{\partial z^\alpha} \tag{VI.73}$$

Hence for fixed values of C^α and K^α we have the n-symplectic rank 1 equation on $(L_\pi E, d\hat{\theta}_L)$

$$d(C^\beta (\Delta u)_\sigma^\alpha h_\beta^\sigma) + K^\alpha) = -(C^\beta \frac{\partial}{\partial z^\beta})\lrcorner\ d\theta_L^\alpha \tag{VI.74}$$

Because K^α are constants, this equation can be rewritten as

$$C^\beta d((\Delta u)_\sigma^\alpha h_\beta^\sigma)) = -(C^\beta \frac{\partial}{\partial z^\beta})\lrcorner\ d\theta_L^\alpha \tag{VI.75}$$

We recall that in section IV we observed that the CHP 1-forms $\theta_L^\alpha = (\Delta u)_\sigma^\alpha h_\beta^\sigma dz^\beta$ define the physical type 1-1 tensor fields on $J^1\pi$ given in equations IV.43 - IV.46. The part $\hat{F} = \left((C^\beta (\Delta u)_\sigma^\alpha h_\beta^\sigma)\right)\ r_\alpha$ of the rank 1 observable VI.72 will thus correspond to linear combinations of these type 1-1 tensor fields on $J^1\pi$; that is linear combinations of field velocities, momentum and energy-momentum tensors.

Remark: Notice that on Q_L the Hamiltonian vector field for the observable $\hat{f} = (C^\alpha \pi_\alpha^\beta + K^\beta)\hat{r}_\beta$ is $C^\alpha \frac{\partial}{\partial \bar{z}^\alpha}$, independent of the constants K^α. In fact the kernel of the mapping $HF^1 \longrightarrow HV^1$ is the set of observables $K^\alpha \hat{r}_\alpha$ where K^α are all constants. Thus the mapping from the Lie algebra of rank-1 observables to the Lie algebra of rank-1 Hamiltonian vector fields is not an isomorphism. In [5] it was shown that this kernel for rank 1 observables can be removed by lifting the theory to the bundle of affine frames of E, thereby obtaining the desired isomorphism from observables to the vector fields that will serve as the Hamiltonian operators in a geometric quantization theory.

VII Conclusions

The formulation of an n-symplectic algebra of observables for a covariant Lagrangian field theory set forth in this paper lays the foundations for a Kostant-Souriau geometric quantization theory of fields. The paper has three main parts. Part one is section II of this paper, in which we developed the n-symplectic Poisson and graded Poisson algebras of observables defined on an arbitrary n-symplectic manifold. These algebras paralleled the algebras presented in [3] for the special case of the bundle of linear frames of an n-dimensional

manifold. In Part two of the paper, consisting of sections III - IV, we set up the n-symplectic covariant field theory on $L_\pi E$ for a Lagrangian field theory. The theory includes the definition of an n-symplectic Legendre transformation, and the subsequent definition of the CHP 1-forms θ_L^α as the pull-back, under the Legendre transformation, of the canonical soldering 1-forms on LE. We then showed that $(L_\pi E, d\hat{\theta}_L)$ is an n-symplectic manifold, and set up the equations of n-symplectic reduction on $(Q_L, d(\hat{\theta}|_{Q_L})$ in order to identify the observables on $L_\pi E$. The third part of the paper, section VI, presents a simple application of the theory to the model of a k-tuple of massless scalar fields on Minkowski spacetime. We found that the rank 1 observables contain the translational Killing vectors of the Minkowski spacetime and the Euclidean fibers. When pulled back to $L_\pi E$ these observables correspond to linear combinations of the field velocities, momentum and energy-momentum tensors on $J^1\pi$.

We have pointed out that in the general theory on $L_\pi E$ the mapping from the Lie algebra of observables to the Lie algebra of Hamiltonian vector fields is not an isomorphism, the kernel of the mapping being the set of all constant $\mathbf{R}^n$-valued functions on $L_\pi E$. This is the analogue of what occurs on a symplectic manifold. In a geometric quantization theory one needs an isomorphism in order to geometrize the Dirac quantization rules. In the present case one can establish such an isomorphism by lifting the theory to an appropriate affine frame bundle using the results in reference [5] where it was shown how to establish this isomorphism for the rank-1 n-symplectic algebras. This result was extended by Cartin to all n-symplectic observables in reference [24]. We have therefore arrived at the frontiers of a geometric quantization theory of fields based on n-symplectic geometry.

VIII   Appendix: Proof of the Jacobi identity for theorem II.9

Let $\hat{X}_{\hat{f}}^I$, $\hat{X}_{\hat{g}}^J$ and $\hat{X}_{\hat{h}}^K$ denote arbitrary sets of representatives of the equivalence classes of Hamiltonian vector fields determined by $\hat{f} \in SHF^p$, $\hat{g} \in SHF^q$ and $\hat{h} \in SHF^r$ respectively, where I, J, K denote the multiindices $I = \alpha_2\alpha_3\ldots\alpha_p$, $J = \alpha_{p+1}\alpha_{p+2}\ldots\alpha_{p+q-1}$ and $K = \alpha_{p+q}\alpha_{p+q+1}\ldots\alpha_{p+q+r-2}$. Then by using $d\hat{\omega} = 0$ and the standard identity for evaluating $d\omega(X, Y, Z)$ for ω a 2-form we obtain

$$
\begin{aligned}
0 &= 3d\omega^{(\alpha}(\hat{X}_{\hat{f}}^I, \hat{X}_{\hat{g}}^J, \hat{X}_{\hat{h}}^{K)}) \\
&= \hat{X}_{\hat{f}}^{(I}\omega^\alpha(\hat{X}_{\hat{g}}^J, \hat{X}_{\hat{h}}^{K)}) + \hat{X}_{\hat{g}}^{(J}\omega^\alpha(\hat{X}_{\hat{h}}^K, \hat{X}_{\hat{f}}^{I)}) + \hat{X}_{\hat{h}}^{(K}\omega^\alpha(\hat{X}_{\hat{f}}^I, \hat{X}_{\hat{g}}^{J)}) \\
&\quad - \omega^{(\alpha}([\hat{X}_{\hat{f}}^I, \hat{X}_{\hat{g}}^J], \hat{X}_{\hat{h}}^{K)}) - \omega^{(\alpha}([\hat{X}_{\hat{h}}^K, \hat{X}_{\hat{f}}^I], \hat{X}_{\hat{g}}^{J)}) - \omega^{(\alpha}([\hat{X}_{\hat{g}}^J, \hat{X}_{\hat{h}}^K], \hat{X}_{\hat{f}}^{I)})
\end{aligned}
$$

$$(VIII.76)$$

Equations (II.9) and (II.20) can be combined to yield the identity

$$\omega^{(\alpha}\left(\hat{X}^I_{\hat{f}}, \hat{X}^{J)}_{\hat{g}}\right) = \frac{1}{2p!\,q!}\{\hat{f},\hat{g}\}^{\alpha IJ} \tag{VIII.77}$$

Using this formula and formula (II.23) in (VIII.76) we obtain

$$
\begin{aligned}
0 \;=\; & \hat{X}^{(I}_{\hat{f}}(\frac{1}{2q!r!}\{\hat{g},\hat{h}\}^{JK\alpha)}) + \hat{X}^{(J}_{\hat{h}}(\frac{1}{2p!q!}\{\hat{f},\hat{g}\}^{IJ\alpha)}) + \hat{X}^{(K}_{\hat{g}}(\frac{1}{2p!r!}\{\hat{h},\hat{f}\}^{KI\alpha)}) \\
& - \frac{(p+q-1)!}{p!q!}\omega^{(\alpha}(X^{IJ}_{\{\hat{f},\hat{g}\}}, \hat{X}^{K)}_{\hat{h}}) - \frac{(p+r-1)!}{p!r!}\omega^{(\alpha}(X^{KI}_{\{\hat{h},\hat{f}\}}, \hat{X}^{J)}_{\hat{g}}) - \frac{(q+r-1)!}{q!r!}\omega^{(\alpha}(X^{JK}_{\{\hat{g},\hat{h}\}}, \hat{X}^{I)}_{\hat{f}})
\end{aligned}
$$

Next we use the definition (II.20) in the first three terms and formula (VIII.77) in the last three terms to obtain

$$
\begin{aligned}
0 \;=\; & \frac{1}{2p!q!r!}\{\hat{f},\{\hat{g},\hat{h}\}\}^L \;+\; \frac{1}{2p!q!r!}\{\hat{h},\{\hat{f},\hat{g}\}\}^L \;+\; \frac{1}{2p!q!r!}\{\hat{g},\{\hat{h},\hat{f}\}\}^L \\
& - \frac{(p+q-1)!}{p!q!}(\frac{1}{2(p+q-1)!r!}\{\{\hat{f},\hat{g}\},\hat{h}\}^L) - \frac{(p+r-1)!}{p!r!}(\frac{1}{2(p+r-1)!q!}\{\{\hat{h},\hat{f}\},\hat{g}\}^L) \\
& - \frac{(q+r-1)!}{q!r!}(\frac{1}{2(q+r-1)!p!}\{\{\hat{g},\hat{h}\},\hat{f}\}^L)
\end{aligned}
$$

where the multi-index L denotes (αIJK). Cancelling the common factor $\frac{1}{p!q!r!}$ we obtain

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\{\hat{f},\{\hat{g},\hat{h}\}\}^L + \frac{1}{2}\{\hat{h},\{\hat{f},\hat{g}\}\}^L + \frac{1}{2}\{\hat{g},\{\hat{h},\hat{f}\}\}^L - \frac{1}{2}\{\{\hat{f},\hat{g}\},\hat{h}\}^L - \frac{1}{2}\{\{\hat{h},\hat{f}\},\hat{g}\}^L - \frac{1}{2}\{\{\hat{g},\hat{h}\},\hat{f}\}^L \\
\;=\; & \{\hat{f},\{\hat{g},\hat{h}\}\}^L + \{\hat{h},\{\hat{f},\hat{g}\}\}^L + \{\hat{g},\{\hat{h},\hat{f}\}\}^L
\end{aligned}
$$

Hence the n-symplectic bracket defined in (II.20) obeys the identity of Jacobi. ∎

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