The Affine Geometry of the Lanczos H-tensor Formalism *

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ABSTRACT

We identify the fiber-bundle-with-connection structure that underlies the Lanczos H-tensor formulation of Riemannian geometrical structure. Motivated by the Lanczos Lagrangian that includes the tensorial constraint that the linear connection be the Levi-Civita connection of the metric tensor, we consider linear connections to be type (1,2) affine tensor fields. We sketch the structure of the appropriate fiber bundle that is needed to describe the differential geometry of such affine tensors, namely the affine frame bundle $A^1_2 M$ with structure group $A^1_2(4) = GL(4) \circledS T^1_2\mathbb{R}^4$ over spacetime M. Generalized affine connections on this bundle are in 1-1 correspondence with pairs (Γ, K) on M, where the $gl(4)$-component Γ denotes a linear connection and the $T^1_2\mathbb{R}^4$-component K is a type (1,3) tensor field on M. Once an origin $\hat{\Gamma}_0$ is chosen in the affine space of linear connections one may define the gauge components of any other linear connection $\hat{\Gamma}$ relative to $\hat{\Gamma}_0$, and these gauge components correspond to type (1,2) tensor fields on M. Taking the origin to be the Levi-Civita connection $\hat{\Gamma}_g$ of the metric tensor g, and taking the pair $(\{{}^{\mu}_{\nu\kappa}\}, R_{\mu\nu\kappa}{}^{\lambda})$ as the generalized Riemannian affine connection, we show that the Lanczos H-tensor arises from a gauge fixing condition on this geometrical structure. The resulting translation gauge, the **Lanczos gauge**, is invariant under the transformations found earlier by Lanczos, and these transformations are identified as elements of a subgroup of the translation group $\{I\} \circledS T^1_2\mathbb{R}^4$. The other Lanczos variables $Q_{\mu\nu}$ and q are constructed in terms of the translational component of the generalized affine connection in the Lanczos gauge. To complete the geometric reformulation we reconstruct the Lanczos Lagrangian completely in terms of affine invariant quantities. The essential field equations derived from our $A^1_2(4)$-invariant Lagrangian are the Bianchi and Bach-Lanczos identities for four-dimensional Riemannian geometry. We also show that the field equations based on the generalized translational curvature that are the analogs of the affine field equations of the $P(4) = O(1,3) \circledS \mathbb{R}^{4*}$ unified theory of gravitation and electromagnetism are equivalent to the field equations of Yang's source-free gravitational gauge theory of the non-integrable phase factor.

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1. Introduction

The classical view of the structure of the four-dimensional Riemannian geometry that underlies relativistic theories is that of a hierarchy of geometrical objects constructed from the fundamental Riemannian metric tensor field $g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$. The metric tensor itself serves to define lengths and angles in the tangent spaces of the spacetime manifold, but when one considers the correlation of lengths and angles at different points of spacetime then the geometrical object of interest is the Levi-Civita linear connection, or Christoffel symbols $\{{}^{\mu}_{\nu\kappa}\}$, constructed from g and its first partial derivatives. In order to determine whether or not the parallel transport defined by this Riemannian connection is path dependent one turns to the Riemannian curvature tensor $R_{\mu\nu\kappa}{}^{\lambda}$ constructed from $\{{}^{\mu}_{\nu\kappa}\}$ and its first partial derivatives. This structural hierarchy can be represented conveniently by the diagram

$$g_{\mu\nu} \xrightarrow{\partial_\kappa} \{{}^{\mu}_{\nu\kappa}\} \xrightarrow{\partial_\kappa} R_{\mu\nu\kappa}{}^{\lambda} \quad . \tag{1.1}$$

The modern version of this classical structure has as its arena the bundle of linear frames LM of the spacetime manifold M. The metric tensor of spacetime defines a unique tensorial $\mathbb{R}^4 \otimes \mathbb{R}^4$-valued function on LM that reduces the structure group $GL(4,\mathbb{R})$ to the group $O(1,3)$ and yields the subbundle OM of orthonormal linear frames. The Levi-Civita connection itself may be viewed as the unique torsion free $gl(4)$-valued connection 1-form ω_g on LM that itself reduces to OM, and the Riemannian curvature tensor $R_{\mu\nu\kappa}{}^{\lambda}$ lifts to the curvature 2-form Ω_g defined by ω_g. Indeed, this well-known fundamental fiber bundle characterization of classical Riemannian geometry is well-understood.

To further dissect the structure of Riemannian geometries one must introduce additional ideas that in turn lead to the decomposition of one or more of the three basic geometric objects in (1.1) above. In 1918 Weyl [1] considered conformal transformations $g(p) \to \bar{g}(p) = e^{\sigma(p)} g(p)$ of the metric tensor and found the (Weyl) conformal curvature tensor

$$C_{\mu\nu\kappa}{}^{\lambda} = R_{\mu\nu\kappa}{}^{\lambda} - 2 g_{[\mu[\kappa} L_{\nu]\alpha]} g^{\alpha\lambda} \quad , \quad L_{\mu\nu} = -R_{\mu\nu} + (\frac{1}{6}) g_{\mu\nu} R \tag{1.2}$$

which remains invariant under conformal transformations.

The conformal curvature tensor $C_{\mu\nu\kappa}{}^{\lambda}$ can also be introduced algebraically. In 1956 Géhéniau and Debever [2] used the Hodge dual operation (see the Appendix) to obtain the following algebraic decomposition of the curvature tensor

$$R_{\mu\nu\kappa\lambda} = C_{\mu\nu\kappa\lambda} + E_{\mu\nu\kappa\lambda} - (\frac{1}{12}) g_{\mu\nu\kappa\lambda} R \quad , \tag{1.3}$$

where $g_{\mu\nu\kappa\lambda} = 2 g_{\mu[\kappa} g_{\lambda]\nu}$ and $E_{\mu\nu\kappa\lambda} = -g_{\mu\nu\alpha[\lambda}(R^{\alpha}_{\kappa]} - (\frac{1}{4}) R \delta^{\alpha}_{\kappa]})$. The tensors C and E satisfy

$$\begin{aligned} C_{\mu\nu\kappa\lambda} &= -\overset{*}{C}{}^{*}_{\mu\nu\kappa\lambda} \\ E_{\mu\nu\kappa\lambda} &= \overset{*}{E}{}^{*}_{\mu\nu\kappa\lambda} \end{aligned} \tag{1.4}$$

and are thus anti-self-double-dual and self-double-dual, respectively.

There exists another classical characterization of the geometrical structure associated with the Weyl conformal curvature tensor, namely the third-order potential structure introduced by Lanczos [3] in 1962. Lanczos considered the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left(R_{\mu\nu\alpha\beta}\overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta} + H_{\mu\nu\alpha}\nabla_\beta \overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta} + P^{\nu\alpha}_{\mu}(\Gamma^{\mu}_{\nu\alpha} - \{{}^{\mu}_{\nu\alpha}\}) + \rho^{\mu\nu}(R_{\mu\nu} - F(\Gamma)_{\mu\nu})\right) \quad , \tag{1.5}$$

where $F(\Gamma)_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - (\frac{1}{2})(\partial_\mu \Gamma^\alpha_{\alpha\nu} + \partial_\nu \Gamma^\alpha_{\alpha\mu}) + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}$. Lanczos had already shown [4] in 1938 that variation of the first term $\mathcal{L}_1 = \sqrt{-g}\left(R_{\mu\nu\alpha\beta}\overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta}\right)$, the Pontrjagin density , with respect to the metric tensor and connection vanishes identically. This is now known to be related to the fact that $R_{\mu\nu\alpha\beta}\overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta}$ is equivalent to the four-form defining the first Pontrjagin class[3] $p_1(TM)$ of the tangent bundle, and $p_1(TM)$ is independent of the connection used in its construction. Lanczos included the remaining terms in (1.5) as constraints so that he could vary the metric tensor $g_{\mu\nu}$, linear connection $\Gamma^\mu_{\nu\kappa}$, and the (double-dual) curvature tensor $\overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta}$ independently. The term involving the Lagrange multiplier $H_{\mu\nu\alpha}$ is the constraint to the Riemannian Bianchi Identity $\nabla_\beta \overset{*}{R}{}^{\mu\nu\overset{*}{\alpha}\beta} = 0$, the third term constrains the linear connection to be the Levi-Civita connection, while the last constraint term gives the definition of $R_{\mu\nu}$ in terms of the connection Γ. Lanczos showed that the Euler-Lagrange equations derived from (1.5) imply that the curvature tensor $R_{\mu\nu\lambda\kappa}$ and the conformal curvature tensor $C_{\mu\nu\lambda\kappa}$ may be expressed as[4]

$$R_{\mu\nu\lambda\kappa} = -4[g_{\mu\lambda}(Q_{\nu\kappa} - q g_{\nu\kappa})] - 4[\nabla_\mu H_{\lambda\kappa\nu}] \quad , \tag{1.6}$$

$$C_{\mu\nu\lambda\kappa} = -4_C[\nabla_\mu H_{\lambda\kappa\nu}] \quad , \tag{1.7}$$

where the tensor $Q_{\mu\nu}$ and scalar $(\frac{1}{4})q$ are the trace-free and trace parts of $\rho_{\mu\nu}$, respectively:

$$\rho_{\mu\nu} = Q_{\mu\nu} + q g_{\mu\nu} \quad , \qquad Q^\mu_\mu = 0 \quad . \tag{1.8}$$

The square brackets used in (1.6) and (1.7) denote symmetrization processes that are discussed in the Appendix. The essential point is that the square bracket used in (1.6), when applied to a rank 4 covariant tensor $A_{\mu\nu\kappa\lambda}$, yeilds a new rank 4 covariant tensor $[A_{\mu\nu\kappa\lambda}]$ that has the same symmetries as does the Riemann curvature tensor. The square bracket with a presubscript of C used in (1.7) adds the additional condition that the resulting tensor ${}_C[A_{\mu\nu\kappa\lambda}]$ be trace-free in all indices with respect to the Riemannian metric tensor g. Hence ${}_C[A_{\mu\nu\kappa\lambda}]$ has the same symmetries as does the Weyl conformal curvature tensor $C_{\mu\nu\kappa\lambda}$.

Lanczos also showed [3] that the left-hand-sides of both (1.6) and (1.7) remain invariant under the combined "gauge type" transformations

$$H_{\mu\nu\lambda} = H'_{\mu\nu\lambda} + (V_\mu g_{\nu\lambda} - V_\nu g_{\mu\lambda}) \tag{1.9a}$$

$$Q_{\mu\nu} = Q'_{\mu\nu} + (\nabla_\mu V_\nu + \nabla_\nu V_\mu - (\frac{1}{2})g_{\mu\nu}\nabla_\lambda V^\lambda) \tag{1.9b}$$

$$q = q' - (\frac{1}{2})\nabla_\mu V^\mu \tag{1.9c}$$

where V_μ is an arbitrary vector. The form of $C_{\mu\nu\lambda\kappa}$ given in (1.7) where it is constructed from first derivatives of $H_{\lambda\kappa\nu}$ shows that $H_{\lambda\kappa\nu}$ may in some sense be considered as a "potential" for the Weyl conformal curvature tensor[5] . We point out that in Ref. 3 Lanczos did not prove the existence of a third order potential $H_{\lambda\kappa\nu}$ for the Weyl conformal tensor of an arbitrary spacetime manifold; rather, he derived field equations for such potentials. Since the Euler-Lagrange equations derived from a variational principle need not be consistent, an existence theorem was needed. Bampi and Caviglia [9] proved that one may always solve the Lanczos field equations for a potential $H_{\lambda\kappa\nu}$ for the Weyl conformal curvature tensor in a four-dimensional analytic spacetime.

Until now the precise sense in which the Lanczos tensor $H_{\lambda\kappa\nu}$ is a potential for the Weyl conformal curvature tensor has not been identified. Moreover the apparent "gauge transformations" (1.9) have not been explained in a gauge theoretic way, nor has the "gauge group" been explicitly identified, and up to now no fiber bundle characterization of this H-tensor structure for Riemannian geometries has been developed. In this paper we show that the theory of generalized affine connections on a certain affine frame bundle $A^1_2M = LM \times_{GL(4)} A^1_2(4)$ is the fiber bundle structure that underlies the Lanczos H-tensor formalism. For a four-dimensional spacetime manifold M the structure group of A^1_2M is the semi-direct product affine group $A^1_2(4) = GL(4) \circledS T^1_2\mathbb{R}^4$. In this setting we will show that the Lanczos theory is a special case of the **theory of the geometry of linearly connected geometries**. That is to say, we will show that when one considers linear connections as affine tensors and introduces a particular generalized affine connection so as to be able to differentiate these affine tensors covariantly, then the Lanczos H-tensor arises in a natural way from a gauge fixing condition imposed on the generalized affine connection.

The structure of the paper is as follows. In Section 2 we present a brief survey of the relevant facts concerning generalized affine connnections for affine tensors of type (1,2). This will allow us to treat the derivatives of linear connections, thought of as affine tensors, in a covariant way. Such a generalized affine connection may be specified by a pair $(\Gamma, {}^{\hat{\Gamma}_0}K)$ on a manifold M, where Γ denotes a linear connnection and ${}^{\hat{\Gamma}_0}K$ is a type (1,3) tensor field on M representing the translational component of the connection. When we specialize to Riemannian geometry we consider the generalized **Riemannian affine connection** specified by the pair (Γ_g, R_g), where Γ_g is the Levi-Civita connection of the metric g and R_g is the Riemann curvature tensor of g.

In Sections 3 and 4 we examine the Lanczos H-tensor formulation of Riemannian geometrical structure using the formalism developed in Section 2. In Section 3 we show that the Lanczos variables $Q_{\mu\nu}$ and q that appear in (1.6)-(1.9) may be constructed in a natural way from the Riemannian affine connection $(\{^{\,\mu}_{\nu\kappa}\}, R_{\mu\nu\kappa}{}^{\lambda})$. The gauge transformations (1.9b) and (1.9c) then follow from the formula for the transformation of a generalized affine connection. Then in Section 4 we show that the Lanczos expression (1.7) for the conformal tensor built from derivatives of the tensor $H_{\mu\nu\kappa}$ follows from a gauge-fixing condition on the generalized affine connection $(\{^{\,\mu}_{\nu\kappa}\}, R_{\mu\nu\kappa}{}^{\lambda})$. The resulting **Lanczos gauge** is unique up to transformations of the form (1.9). In this sense the subgroup of the translations defined by (1.9) defines the **Lanczos symmetry subgroup** of $T^1_2\mathbb{R}^4$.

In Section 5 we reconstruct the Lanczos Lagrangian (1.5) from affine invariant quantities. The Lanczos variational variables $(g_{\mu\nu}, \Gamma^\mu_{\nu\kappa}, R^{\overset{*}{\mu}\nu\overset{*}{\alpha}\beta})$ are replaced by the variables $(g_{\mu\nu}, (\Gamma^\mu_{\nu\kappa}, K_{\mu\nu\alpha}{}^{\beta}))$. Thus the Riemannian metric tensor is retained, and $\Gamma^\mu_{\nu\kappa}$ and $R^{\overset{*}{\mu}\nu\overset{*}{\alpha}\beta}$

are replaced by a generalized affine connection. The affine field equations derived from our Lagrangian yield the Bianchi and Bach-Lanczos identities [4,11] for 4-dimensional Riemannian geometry as well as the Lanczos potential structure equations (1.6) and (1.7). Finally in Section 6 we present a summary of our results together with a comparison of the basic structure of the theory with that of the recently introduced $P(4) = O(1,3) \circledS \mathbb{R}^{4*}$ affine unified theory of gravitation and electromagentism [12]. We argue that the natural $A_2^1(4)$ analogue of the P(4) theory yeilds Yang's [13] source-free gravitational gauge theory of the non-integrable phase factor. In the Appendix we discuss certain technical details that are needed throughout the paper.

2. The Affine Geometry of Linear Connections.

If one restricts attention to tensor geometry on a manifold M then the basic structure needed to introduce a covariant differentiation of tensor fields is a linear connection 1-form ω on the bundle of linear frames LM of M, or equivalently connection coefficients

$$\Gamma = (\Gamma^{\mu}_{\nu\kappa}) \tag{2.1}$$

of ω on open sets of the base manifold M. Given Γ and any tensor field T on M one can differentiate T in a tensorial way using the covariant derivative

$$\nabla^{\Gamma}(T) \tag{2.2}$$

of T with respect to Γ. Moreover, the fundamental tensor formed from derivatives of Γ is the curvature 2-form

$$\Omega = d\omega + \omega \wedge \omega \tag{2.3}$$

on LM, and the corresponding curvature tensor

$$R_{\mu\nu\lambda}{}^{\kappa} = 2\partial_{[\mu}\Gamma^{\kappa}_{\nu]\lambda} + 2\Gamma^{\kappa}_{[\mu|\rho|}\Gamma^{\rho}_{\nu]\lambda} \tag{2.4}$$

on the base manifold M.

This structure is sufficient for any theory that employs tensor geometry relative to a fixed (perhaps arbitrary) linear connection. If one considers a theory where the connection itself is a variable then one needs to deal with connections in a covariant way. Unfortunately, the components $\Gamma^{\mu}_{\nu\kappa}$ of the connection do not, however, themselves form the components of a tensor field. Connection 1-forms ω on LM are pseudo-tensorial rather than tensorial, and this is of course the reason for the inhomogeneous transformation law for the connection coefficients $\Gamma^{\mu}_{\nu\kappa}$. On the other hand it is well known that the set of all linear connections $\mathcal{C}$ on a manifold M forms an affine space. The basic feature of this affine structure is that the difference between two linear connections $(\Gamma_2)^{\mu}_{\nu\lambda}$ and $(\Gamma_1)^{\mu}_{\nu\lambda}$ is a tensor $T_{\nu\lambda}{}^{\mu}$:

$$(\Gamma_2)^{\mu}_{\nu\lambda} - (\Gamma_1)^{\mu}_{\nu\lambda} = T_{\nu\lambda}{}^{\mu} \quad . \tag{2.5}$$

Put another way, any type (1,2) tensor field $T^{\mu}_{\nu\lambda}$ can be added to a linear connection $(\Gamma_1)^{\mu}_{\nu\lambda}$ to obtain another linear connection $(\Gamma_2)^{\mu}_{\nu\lambda}$:

$$(\Gamma_2)^{\mu}_{\nu\lambda} = (\Gamma_1)^{\mu}_{\nu\lambda} + T_{\nu\lambda}{}^{\mu} \quad . \tag{2.6}$$

But notice that the addition on the right hand side of (2.6) is the addition of two distinctly different types of geometric objects. Moreover, one notices that the addition of two linear connections does not produce a third linear connection. The underlying mathematical structure is that of an **affine space**[6] in which addition of points is not defined (sum of two linear connections not a linear connection), but in which an operation of subtraction that yields a "vector" is defined (difference of two linear connections produces a type (1,2) tensor). Thus the set of all linear connections on a manifold M is an infinite dimensional affine space modeled on the infinite dimensional vector space of type (1,2) tensor fields on M. We now suppress indices and rewrite (2.6) as

$$\hat{\Gamma}_2 = \hat{\Gamma}_1 \oplus T \tag{2.7}$$

where the symbol $\oplus$ signifies that the meaning of this equation is contained in the equation (2.6). As this equation indicates we will use the notational convention of putting a hat over a connection when considering it as an "affine tensor".

The geometrical arena for type (1,2) affine tensors is the affine frame bundle A^1_2M of type (1,2). Points in the bundle space of A^1_2M are affine frames, namely triples (p, e_μ, T) where (e_μ) is a linear frame for the tangent space T_pM and where T is a type (1,2) tensor at $p \in M$, the "origin" of the affine frame. The group of A^1_2M is the affine group of type (1,2) $A^1_2(4) = GL(4) \circledS T^1_2\mathbb{R}^4$. One can build up the theory of generalized affine connections for A^1_2M along the lines followed by Kobayashi and Nomizu [15]. Here, however, we present only the relevant facts in local spacetime coordinates that will be needed to reformulate the Lanczos theory in terms of affine tensors.

From the point of view of gauge theory one may consider the expression (2.7) as defining T as the gauge components of Γ_2 with respect to the gauge (origin) Γ_1. Moreover, one knows that in a gauge theory one needs a gauge connection in order to bring covariance to expressions involving derivatives of the gauge components of geometrical objects. In the affine theory of linear connections as affine tensors one thus needs a generalized affine connection to deal with objects with both ordinary tensorial properties plus the new properties associated with the affine transformation law (2.7). We introduce a generalized affine connection for such objects as follows.

Choose an origin $\hat{\Gamma}_0 \in \mathcal{C}$ and a linear frame field (e_μ) on an open set in M. Such a pair $(e_\mu, \hat{\Gamma}_0)$ will be referred to as an **affine frame** field for affine tensors of type (1,2). We are thus identifying $\hat{\Gamma}_0$ with the zero tensor field of type (1,2). Then define the components $(\Gamma^\lambda_{\mu\nu}, {}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa)$ of a generalized affine connection with respect to this affine frame by

$$D_\mu(e_\nu) \equiv \nabla_\mu(e_\nu) = \Gamma^\lambda_{\mu\nu} e_\lambda \tag{2.8a}$$

$$D_\mu((\hat{\Gamma}_0)^\kappa_{\nu\lambda}) = {}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa \tag{2.8b}$$

This definition of a generalized affine connection is a Kozul-type definition. In (2.8a) ∇_μ denotes the linear covariant derivative with respect to the connection Γ.

We will refer to $\Gamma^\lambda_{\mu\nu}$ and ${}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa$ as the linear and translational components, respectively, of the generalized affine connection. The linear component $\Gamma^\lambda_{\mu\nu}$ are the coefficients of a linear connection while the translational component ${}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa$ is a type (1,3) tensor field on M. The left superscript on ${}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa$ indicates the explicit dependence of the translational component of the connection on the choice of origin of the affine frame.

Affine frame fields are related by $A^1_2(4)$ gauge transformations. Let $(g^\mu_\nu, T_{\beta\gamma}{}^\alpha)$ be a point dependent $A^1_2(4)$-valued function on M. Then the gauge transformation formula is

$$(e_\mu, \hat{\Gamma}_0) \longrightarrow (e'_\mu, \hat{\Gamma}'_0) = (e_\mu, \hat{\Gamma}_0) \cdot (g^\mu_\nu, T_{\beta\gamma}{}^\alpha) = (e_\mu g^\mu_\nu, \hat{\Gamma}_0 + T) \tag{2.9}$$

where $T = T_{\beta\gamma}{}^\alpha e_\alpha \otimes e^\beta \otimes e^\gamma$. Under such a gauge transformation the linear component of the affine connection transforms in the standard way. On the otherhand for the translational component one finds

$${}^{\hat{\Gamma}_0 \oplus T}K_\mu = {}^{\hat{\Gamma}_0}K_\mu + \nabla_\mu(T) \quad . \tag{2.10}$$

This is the **gauge transformation formula** for the translational components of a generalized affine connection under the change of origin $\hat{\Gamma}_0 \longrightarrow \hat{\Gamma}_0 \oplus T$.

The curvature of the generalized affine connection $(\Gamma^\lambda_{\mu\nu}, {}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^\kappa)$ can be constructed using standard techniques. We denote this curvature by the pair $(R, {}^{\hat{\Gamma}_0}\Phi)$ where

$$R = (R_{\mu\nu\lambda}{}^\kappa) \tag{2.11}$$

is the linear curvature tensor of the linear connection $\Gamma^\mu_{\nu\lambda}$ given in (2.4) above, and

$$({}^{\hat{\Gamma}_0}\Phi)_{\mu\nu\alpha\beta}{}^\kappa = 2\nabla_{[\mu}({}^{\hat{\Gamma}_0}K)_{\nu]\alpha\beta}{}^\kappa + 2S_{\mu\nu}{}^\sigma({}^{\hat{\Gamma}_0}K)_{\sigma\alpha\beta}{}^\kappa \tag{2.12}$$

are the components of the translational curvature with respect to the origin Γ_0. The tensor $S_{\mu\nu}{}^\sigma$ in (2.12) is the torsion of the linear connection Γ.

We note for later applications that if Γ is any linear connection on M and R is its curvature tensor, then R is a type (1,3) tensor field. Hence we may infer that the pair (Γ, R) defines a generalized affine connection for type (1,2) affine tensors. In this setting we note that the usual rules for exterior covariant differentiation of the affine tensor $\hat{\Gamma}_1$ with respect to the affine connection (Γ, R) yields the formula

$$D_\mu(\hat{\Gamma}_1)^\nu_{\lambda\kappa} = \nabla_\mu(T_{\lambda\kappa}{}^\nu) + R_{\mu\lambda\kappa}{}^\nu \quad . \tag{2.13}$$

Here $T_{\lambda\kappa}{}^\nu = (\Gamma_1)^\nu_{\lambda\kappa} - \Gamma^\nu_{\lambda\kappa}$ is the base space form of the tensorial difference form defined by the two connections Γ_1 and Γ. Equation (2.13) is equivalent to the formula $D^\omega(\omega_1) = D^\omega(\omega_1 - \omega) + \Omega$ one would obtain on LM for the exterior covariant derivative of the connection 1-form ω_1 of Γ_1 with respect to the linear connection 1-form ω of Γ. Note, however, that the second term on the right in (2.13), which represents the curvature of ω, is **independent** of Γ_1 and $T_{\nu\lambda}{}^\mu$, and this non-linear term shows that the proper setting for covariant derivatives of connections is generalized affine geometry.

A Riemannian metric defines a Levi-Civita connection Γ_g and a Riemannian curvature tensor R_g on M, and this pair (Γ_g, R_g) thus defines a generalized affine connection for type (1,2) affine tensors, which we will refer to as a **Riemannian affine connection**. Since the torsion of the Levi-Civita connection is trivial the translational curvature in this case reduces to the simple form

$$({}^{\hat{\Gamma}_0}\Phi)_{\mu\nu\alpha\beta}{}^\kappa = 2\nabla_{[\mu}({}^{\hat{\Gamma}_0}K)_{\nu]\alpha\beta}{}^\kappa \tag{2.14}$$

Finally we note that in Section 5 we will need a particular double dual curvature constructed from the translational component of the full affine curvature in an arbitrary gauge $\hat{\Gamma}$. We suppose that this curvature is defined on a spacetime manifold (M, g) and we use the associated Hodge dual operator to define an associated rank three tensor $({}^{\hat{\Gamma}}\Phi)^{\overset{*}{\mu}\overset{*}{\nu}\kappa}$ by

$$({}^{\hat{\Gamma}}\Phi)^{\overset{*}{\mu}\overset{*}{\nu}\kappa} = (\frac{1}{3!2!})\eta^{\mu\alpha\beta\gamma}\eta^{\nu\kappa\lambda\rho}({}^{\hat{\Gamma}}\Phi)_{\alpha\beta\gamma\lambda\rho} \quad . \tag{2.15}$$

3. The Lanczos Gauge Transformations as Affine Gauge Transformations

The Lanczos H-tensor formulation of Riemannian geometrical structure arises from a variational principle which is based purely on linear connections and the associated linear differential geometry. As discussed in Section 1, the results of this formalism relate the Riemannian curvature tensor $R_{\alpha\beta\mu}{}^{\nu}$ and the Weyl conformal curvature tensor $C_{\alpha\beta\mu}{}^{\nu}$ in any given Riemannian geometry to the Lagrange multipliers $H_{\alpha\beta}{}^{\mu}$, $Q_{\alpha\beta}$ and q. Furthermore, if certain "gauge" conditions are placed on $H_{\alpha\beta}{}^{\mu}$, the above identifications are unique, up to a very specific type of gauge transformation. In this and the next section we will show that this entire Lanczos formalism may be reformulated in a natural way in terms of the affine geometry of type (1,2) affine tensors.

In this section we concentrate on the Lanczos Lagrange multipliers $Q_{\alpha\beta}$ and q and show that they may be constructed in a natural way from the translational component of a generalized affine connection. The Lanczos variables $Q_{\alpha\beta}$ and q will thus be shown to be affine tensors, and accordingly they will transform inhomogeneously under the translational subgroup of $A^1_2(4)$. In particular the Lanczos gauge transformations given in relations (1.9b) and (1.9c) will be shown to be a special case of general affine gauge transformations.

Throughout this section we assume a Riemannian affine connection (Γ_g, R_g) on a 4-dimensional spacetime (M,g). In Section 2 we have seen that any type (1,2) tensor T induces a translation in the affine space of connections of the form $\hat{\Gamma} \to \hat{\Gamma} \oplus T$. In order to set up an affine model of the Lanczos formalism we first restrict the translations in $A^1_2(4)$ to the subgroup G defined by antisymmetric tensors $A_{\mu\nu\alpha} \stackrel{def}{=} T_{\alpha[\mu\nu]}$. If we make the choice $\Gamma_g \longrightarrow \hat{\Gamma}_g$ for the origin in the affine space of linear connections, where Γ_g is the Levi-Civita connection on M, then a transformation $\hat{\Gamma}_g \to \hat{\Gamma}_g \oplus A$ by elements of the subgroup now takes the form

$$\Gamma^{\nu}_{\alpha\mu} = \{{}^{\nu}_{\alpha\mu}\} + A_{\mu}{}^{\nu}{}_{\alpha} \quad . \tag{3.1}$$

Clearly each linear connection $\Gamma^{\nu}_{\alpha\mu}$ obtained from a translation of the form given in (3.1) is a **metric connection**.

Consider next the translational components ${}^{\hat{\Gamma}'}K_{\mu\nu\alpha\beta}$ of the Riemannian affine connection expressed in any general gauge $\hat{\Gamma}' = \hat{\Gamma}_g \oplus A'$. A further translation of the form $\hat{\Gamma}' \to \hat{\Gamma}' \oplus A$ implies that the translational connection transforms as

$${}^{\hat{\Gamma}'\oplus A}K_{\mu\nu\alpha\beta} = {}^{\hat{\Gamma}'}K_{\mu\nu\alpha\beta} + \nabla_{\mu}A_{\alpha\beta\nu} \quad . \tag{3.2}$$

Using the bracket operation defined in the Appendix we induce the symmetries of the Riemannian curvature tensor on each term in (3.2):

$$[{}^{\hat{\Gamma}'\oplus A}K_{\mu\nu\alpha\beta}] = [{}^{\hat{\Gamma}'}K_{\mu\nu\alpha\beta}] + [\nabla_{\mu}A_{\alpha\beta\nu}] \quad . \tag{3.3a}$$

It is convenient to introduce an associated tensor $\tilde{A}_{\alpha\beta\mu}$ constructed from $A_{\alpha\beta\mu}$ as in (A5) that has the additional symmetry property $\tilde{A}_{[\alpha\beta\mu]} = 0$. Moreover, it follows from (A6) in conjunction with (3.3a) that $[{}^{\hat{\Gamma}'\oplus A}K_{\mu\nu\alpha\beta}] = [{}^{\hat{\Gamma}'\oplus \tilde{A}}K_{\mu\nu\alpha\beta}]$. Hence, when dealing with the bracket operation one may work with $\tilde{A}_{\alpha\beta\mu}$ rather than $A_{\alpha\beta\mu}$. Therefore, instead of (3.3a) we will consider the relation

$$[{}^{\hat{\Gamma}'\oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = [{}^{\hat{\Gamma}'}K_{\mu\nu\alpha\beta}] + [\nabla_{\mu}\tilde{A}_{\alpha\beta\nu}] \tag{3.3b}$$

This equation is the starting point for many of our developments.

In terms of the translational components of the affine connection in any translational gauge $\hat{\Gamma}'$, we define the **generalized Lanczos affine tensors** ${}^{\hat{\Gamma}'}Q_{\nu\alpha}$ and ${}^{\hat{\Gamma}'}q$ by

$$ {}^{\hat{\Gamma}'}Q_{\nu\alpha} \overset{def}{=} \frac{1}{2}\left([{}^{\hat{\Gamma}'}K_{\nu\alpha}] - \frac{1}{4}g_{\nu\alpha}[{}^{\hat{\Gamma}'}K]\right) \tag{3.4a} $$

$$ {}^{\hat{\Gamma}'}q \overset{def}{=} -(\frac{1}{24})[{}^{\hat{\Gamma}'}K] \quad . \tag{3.4b} $$

In the above expressions we have used certain contractions which are discussed in the Appendix.

Since ${}^{\hat{\Gamma}'}Q_{\nu\alpha}$ and ${}^{\hat{\Gamma}'}q$ are constructed directly from the translational component of the affine connection they do not transform as linear tensors. In particular, it follows from (3.3) and (3.4) that under the general translation of origin $\hat{\Gamma}' \to \hat{\Gamma}' \oplus \tilde{A}$ by an arbitrary antisymmetric tensor $\tilde{A}_{\alpha\beta\nu} = \tilde{A}_{[\alpha\beta]\nu}$ the generalized Lanczos affine tensors transform according to the rules

$$ {}^{\hat{\Gamma}'\oplus\tilde{A}}Q_{\nu\alpha} = {}^{\hat{\Gamma}'}Q_{\nu\alpha} - \frac{1}{8}(\tilde{A}_{\nu\alpha} + \tilde{A}_{\alpha\nu}) + \frac{1}{16}g_{\nu\alpha}\tilde{A}^{\beta}_{\beta} \tag{3.5a} $$

$$ {}^{\hat{\Gamma}'\oplus\tilde{A}}q = {}^{\hat{\Gamma}'}q + \frac{1}{48}\tilde{A}^{\beta}_{\beta} \quad , \tag{3.5b} $$

respectively. Here we have used the definition $\tilde{A}_{\nu\alpha} = \nabla_\mu \tilde{A}_{\nu\cdot\alpha}^{\ \mu} - \nabla_\alpha \tilde{A}_{\nu\mu\cdot}^{\ \ \mu}$.

We now want to exhibit a special property of the generalized Lanczos affine tensors ${}^{\hat{\Gamma}}Q_{\mu\nu}$ and ${}^{\hat{\Gamma}}q$. We first consider the specialization of the transformations (3.5) under the subgroup of translations by antisymmetric tensors $\tilde{A}_{\alpha\beta\nu}$ of the form

$$ \tilde{A}_{\alpha\beta\nu} = 4(V_\alpha g_{\beta\nu} - V_\beta g_{\alpha\nu}) \tag{3.6} $$

where V_α is an arbitrary vector. This special form is motivated by the transformation (1.9a) on $H_{\mu\nu\lambda}$ considered by Lanczos. Substituting (3.6) into (3.5a) and (3.5b) we find the specialized transformations

$$ {}^{\hat{\Gamma}}Q_{\nu\alpha} = {}^{\hat{\Gamma}'}Q_{\nu\alpha} + (\nabla_\alpha V_\nu + \nabla_\nu V_\alpha - \frac{1}{2}g_{\alpha\nu}\nabla_\mu V^\mu) \tag{3.7a} $$

$$ {}^{\hat{\Gamma}}q = {}^{\hat{\Gamma}'}q - \frac{1}{2}\nabla_\mu V^\mu \quad , \tag{3.7b} $$

respectively. On the left-hand sides of the above equations we have relabeled the transformed gauge as $\hat{\Gamma} \overset{def}{=} \hat{\Gamma}' \oplus \tilde{A}$, where $\tilde{A}$ is defined in (3.6). We observe that transformations (3.7a) and (3.7b) have precisely the same form as the Lanczos transformations given in (1.9b) and (1.9c). They arise in the present formalism as the transformation laws of the affine tensors ${}^{\hat{\Gamma}}Q_{\mu\nu}$ and ${}^{\hat{\Gamma}}q$, constructed from the translational component of a generalized affine connection, under the restricted translations defined in (3.6) above.

4. The Lanczos H-tensor in Affine Geometry: The Lanczos Gauge

In the last section we showed that the naturally defined affine tensors ${}^{\hat{\Gamma}}Q_{\mu\nu}$ and ${}^{\hat{\Gamma}}q$ have precisely the same transformation properties as do the Lanczos Lagrange multipliers $Q_{\mu\nu}$ and q. In this section we complete the affine reformulation of the Lanczos method. In particular we show that the Lanczos H-tensor arises from a **gauge fixing condition** imposed on a Riemannian affine connection. Moreover, we will show that the entire Lanczos formalism contained in relations (1.6) and (1.7) can also be recovered from the same gauge fixing condition.

For the sake of generality we will initially choose our generalized affine connection to be $(\{{}^{\alpha}_{\beta\mu}\}, {}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta})$, where ${}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta}$ is an arbitrary rank four tensor field. As discussed in the previous section we restrict translations to translations by anti-symmetric tensors $A_{\alpha\beta\mu} = T_{\mu[\alpha\beta]}$. In particular, a translation $\hat{\Gamma}_g \to \hat{\Gamma}_g \oplus A$ induces a transformation of the translational component of the generalized affine connection (see (3.2)) which may be used to obtain (3.3a) and, without loss of generality (3.3b), where we make the identification $\hat{\Gamma}' = \hat{\Gamma}_g$.

The starting point for the rest of this section is thus the fundamental equation (3.3b) with $\hat{\Gamma}' = \hat{\Gamma}_g$. As described in the Appendix the bracket operator can be split into a "conformal" (or trace-free) part ${}_C[\ \ ]$ and a "local" (or trace) part ${}_L[\ \ ]$ defined in (A3) and (A4), respectively. Using these ideas we may decompose (3.3b) as

$$ {}_C[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = {}_C[{}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta}] + {}_C[\nabla_\mu \tilde{A}_{\alpha\beta\nu}] \quad , \tag{4.1} $$

$$ {}_L[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = {}_L[{}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta}] + {}_L[\nabla_\mu \tilde{A}_{\alpha\beta\nu}] \quad . \tag{4.2} $$

Since the object ${}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}$ is part of a generalized affine connection, we may consider gauge-fixing conditions on this part of the connection. In particular, we will investigate the consequences of the specific gauge-fixing condition

$$ {}_C[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = 0 \quad . \tag{4.3} $$

This condition (4.3) in conjunction with (4.1) implies

$$ {}_C[{}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta}] = -{}_C[\nabla_\mu \tilde{A}_{\alpha\beta\nu}] \quad . \tag{4.4} $$

Recently, Bampi and Caviglia [9] have investigated the integrability conditions associated with (4.4) thought of as a partial differential equation for $\tilde{A}_{\mu\nu\kappa}$ and have shown that local solutions of (4.4) always exist. Furthermore they showed that a solution is unique up to addition of a rank three tensor field of the form given in (3.6). The only assumption required for the existence of local solutions of (4.4) is that (M,g) be a four dimensional analytic Riemannian spacetime.

Throughtout the remainder of this section we shall choose the generalized affine connection to be the Riemannian affine connection (Γ_g, R_g) as we did in Section 3. With this assumption we note that since ${}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$, we have ${}_C[{}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta}] = C_{\mu\nu\alpha\beta}$, and thus (4.4) reduces to

$$ C_{\mu\nu\alpha\beta} = -{}_C[\nabla_\mu \tilde{A}_{\alpha\beta\nu}] \quad . \tag{4.5} $$

Relation (4.5) is clearly of the same form as that given in (A7) which arises in the standard Lanczos formalism, and we are thus motivated to introduce the following definition.

Def. #4.1: Let ${}^{\hat{\Gamma}_g}K_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$ and let the tensor field $\tilde{A}_{\alpha\beta\nu}$, defined by the anti-symmetric tensor $A_{\alpha\beta\nu} = -A_{\beta\alpha\nu}$ as in (A5), satisfy ${}_C[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = 0$. Then we define the **generalized Lanczos tensor** $H_{\alpha\beta}{}^{\mu}$ by

$$H_{\alpha\beta\mu} = (\frac{1}{4})\tilde{A}_{\alpha\beta\mu} \quad . \tag{4.6}$$

Note that it follows from the properties of $\tilde{A}_{\alpha\beta\mu}$ that $H_{[\alpha\beta\mu]} = 0$. It does not follow, however, that $H_{\alpha\beta}{}^{\mu}$ also satisfies the other standard Lanczos "gauge" conditions $H_{\alpha\beta}{}^{\beta} = 0$ and/or $\nabla_\mu H_{\alpha\beta}{}^{\mu} = 0$.

Clearly, if we use definition (4.6) the gauge translation of interest can be written as $\hat{\Gamma}_g \to \hat{\Gamma}_g \oplus \tilde{A} = \hat{\Gamma}_g \oplus 4H$, and condition (4.3) becomes ${}_C[{}^{\hat{\Gamma}_g \oplus 4H}K_{\mu\nu\alpha\beta}] = 0$. Furthermore, with this same identification (4.5) reduces precisely to the Lanczos form of $C_{\mu\nu\alpha\beta}$ in terms of $H_{\mu\nu\alpha}$ given in (A.7). The above discussion shows that the generalized Lanczos tensor arises, within the context of this new affine formalism, from a **gauge fixing condition** placed on the translational component of the Riemannian affine connection (Γ_g, R_g) defined by the spacetime metric tensor. We will refer to any translational gauge defined by (4.3) as a **Lanczos gauge**. Such a gauge is unique up to gauge translations by tensors of the form defined in (3.6), and the subgroup of the translation group defined in this way will be called the **Lanczos symmetry group**.

Below we consider two theorems, the first of which is an adaptation of the main results of Bampi and Caviglia [9] to affine geometry while the second theorem represents a new affine version of the Lanczos H-tensor formulation of Riemannian geometrical structure. The general setting for the theorems is as follows. We consider a given four-dimensional analytic Riemannian spacetime manifold M with metric tensor field g, and as above we denote by Γ_g the torsion-free linear connection 1-form defined by g. In the affine space of linear connections we choose an origin by setting $\Gamma_g \to \hat{\Gamma}_g$ as we have done in Section 2. Furthermore, we select the Riemannian affine connection (Γ_g, R_g) determined by Γ_g. Then in local coordinates on M we have the local components of the translational part of the generalized affine connection given by ${}^{\hat{\Gamma}_g}K_{\mu\nu\alpha}{}^{\beta} = R_{\mu\nu\alpha}{}^{\beta}$.

Theorem 4.1: There always exist local solutions $\tilde{A}_{\alpha\beta\mu}$ of the equation

$${}_C[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = 0 \quad , \tag{4.7}$$

where $\tilde{A}_{\alpha\beta\mu}$ is anti-symmetric $\tilde{A}_{\alpha\beta\mu} = \tilde{A}_{[\alpha\beta]\mu}$, and satisfies $\tilde{A}_{[\alpha\beta\mu]} = 0$. Moreover, since this equation is equivalent to

$$C_{\mu\nu\alpha\beta} = -{}_C[\nabla_\mu \tilde{A}_{\alpha\beta\nu}] \quad , \tag{4.8}$$

a particular solution of (4.7) is provided by a generalized Lanczos tensor defined by a solution $\tilde{A}_{\alpha\beta\mu}$ of (4.8).

Theorem 4.2: Suppose that the tensor field $\tilde{A}_{\alpha\beta\mu}$ is anti-symmetric $\tilde{A}_{\alpha\beta\mu} = \tilde{A}_{[\alpha\beta]\mu}$, and suppose that it satisfies $\tilde{A}_{[\alpha\beta\mu]} = 0$ and the gauge-fixing condition

$$_C[^{\hat{\Gamma}_g \oplus \tilde{A}} K_{\mu\nu\alpha\beta}] = 0 \quad . \tag{4.9}$$

Let $H_{\alpha\beta\nu}$ be a generalized Lanczos tensor defined by $\tilde{A}_{\alpha\beta\nu}$. Then

$$C_{\mu\nu\alpha\beta} = -4(_C[\nabla_\mu H_{\alpha\beta\nu}]) \tag{4.10}$$

and

$$R_{\mu\nu\alpha\beta} = -4[\nabla_\mu H_{\alpha\beta\nu}] - 4[g_{\mu\alpha}\left(^{\hat{\Gamma}_g \oplus 4H} Q_{\nu\beta} - (^{\hat{\Gamma}_g \oplus 4H} q) g_{\nu\beta}\right)] \tag{4.11}$$

where $^{\hat{\Gamma}_g \oplus 4H} Q_{\nu\beta}$ and $^{\hat{\Gamma}_g \oplus 4H} q$ are defined in (3.4a) and (3.4b), respectively.

Proof: The gauge-fixing condition (4.9) is a restatement of the condition given in (4.3). Relation (4.10) follows directly from (4.8) of Theorem 4.1 in conjunction with Definition 4.1.

Next we note that relation (4.11) follows from (4.9) by considering the local part of the bracket which appears in (4.2). In particular, given a general $^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}$, expressed in any gauge $\hat{\Gamma}'$, we construct $_L[^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}]$ according to definition (A4). Moreover, we also use definitions (3.4a) and (3.4b) in definition (A4) to rewrite the previous relation in the form

$$_L[^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}] = -4[g_{\mu\alpha}\left(^{\hat{\Gamma}'} Q_{\nu\beta} - (^{\hat{\Gamma}'} q) g_{\nu\beta}\right)] \quad . \tag{4.12}$$

Next write $[^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}] = {}_C[^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}] + {}_L[^{\hat{\Gamma}'} K_{\mu\nu\alpha\beta}]$ and, in particular, if we let $\hat{\Gamma}' = \hat{\Gamma}_g \oplus \tilde{A}$, then (4.9) implies

$$[^{\hat{\Gamma}_g \oplus \tilde{A}} K_{\mu\nu\alpha\beta}] = {}_L[^{\hat{\Gamma}_g \oplus \tilde{A}} K_{\mu\nu\alpha\beta}] \quad . \tag{4.13}$$

Finally, we use (4.12) (with $\hat{\Gamma}' = \hat{\Gamma}_g \oplus \tilde{A}$) in (4.13), and then insert the result into (3.3b). We can now use the fact that $[^{\hat{\Gamma}_g} K_{\mu\nu\alpha\beta}] = R_{\mu\nu\alpha\beta}$ together with Definition 4.1 to rewrite (3.3b) in the form (4.11). ∎

Relation (4.10) clearly has the same form as Lanczos's formula (1.7) for the conformal curvature tensor in terms of his H-tensor. Similarly, relation (4.11) is also of the same form as the analogous relation (1.6) obtained by Lanczos. In the new affine formulation both (4.10) and (4.11) arise directly from the transformation of the translational component of the generalized affine connection, as in (3.2), in conjunction with the single gauge-fixing condition (4.9).

5. An Affine Generalization of the Lanczos Variational Principle

In this section we consider a Palatini type of variational principle which represents an affine generalization of the Lanczos variational principle[7] based on the Langrangian (1.5). We will show that the resulting field equations are the Bianchi and Bach-Lanczos identities [4,11] for four-dimensional Riemannian spacetime geometry.

In our affine variational principle we will vary the $A_2^1(4)$-invariant Langrangian (5.1) with respect to the metric $g_{\mu\nu}$ and a generalized affine connection $(\Gamma^{\mu}_{\lambda\kappa}, {}^{\hat{\Gamma}_0}K_{\mu\nu\lambda}{}^{\kappa})$. We recall that Lanczos added the constraint terms in (1.5) so that he could vary as independent variables the metric $g_{\mu\nu}$, the linear connection $\Gamma^{\mu}_{\nu\lambda}$, and the double-dual curvature tensor $R^{\mu^*\nu\lambda^*\kappa}$. Since the variation of the generalized affine connection corresponds to a variation of the pair $(\Gamma^{\alpha}_{\beta\mu}, {}^{\hat{\Gamma}}K_{\alpha\beta\mu}{}^{\nu})$ on spacetime M, our variation variables would be the same as the Lanczos variables if we would constrain ${}^{\hat{\Gamma}}K^{\mu\nu\lambda\kappa}$ to be $R^{\mu^*\nu\lambda^*\kappa}$. However, we find it more natural to constrain the translational component of the connection to be the Riemann tensor $R_{\mu\nu\lambda}{}^{\kappa}$. To within this difference the Lanczos variables $(\Gamma^{\mu}_{\nu\kappa}, R^{\mu^*\nu\lambda^*\kappa})$ are unified in the generalized affine connection $(\Gamma^{\mu}_{\nu\kappa}, {}^{\hat{\Gamma}}K_{\mu\nu\lambda}{}^{\kappa})$. Our generalization of (1.5) is the $A_2^1(4)$-invariant Lagrangian density

$$L = \sqrt{-g}(\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5) \tag{5.1}$$

where the individual terms $\mathcal{L}_i$ are:

$$\begin{aligned} \mathcal{L}_1 &= (\hat{D}\hat{\Gamma}_0) \cdot {}^*(\hat{D}\hat{\Gamma}_0)^* \\ &= (\frac{1}{4})(\eta^{\alpha\beta\mu\nu}\eta^{\gamma\delta\sigma\rho})g_{\lambda\delta}g_{\rho\kappa}\left(({}^{\hat{\Gamma}_0}K)_{\alpha\beta\gamma}{}^{\lambda}({}^{\hat{\Gamma}_0}K)_{\mu\nu\sigma}{}^{\kappa}\right) \quad , \end{aligned} \tag{5.2}$$

$$\mathcal{L}_2 = B_{\nu\lambda\mu}\left(({}^{\hat{\Gamma}}\Phi)^{\overset{*}{\mu}\overset{*}{\nu}\lambda} - ({}^{\hat{\Gamma}}\tilde{\Phi})^{\overset{*}{\mu}\overset{*}{\nu}\lambda}\right) \quad , \tag{5.3}$$

$$\mathcal{L}_3 = \Sigma^{\alpha\beta\mu}{}_{\nu}\left(({}^{\hat{\Gamma}}K_{\alpha\beta\mu}{}^{\nu} - {}^{\hat{\Gamma}}\tilde{K}_{\alpha\beta\mu}{}^{\nu}) - \overset{0}{R}_{\alpha\beta\mu}{}^{\nu}\right) \quad , \tag{5.4}$$

$$\mathcal{L}_4 = P^{\mu\nu}{}_{\lambda}(\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu} - \{{}^{\lambda}_{\mu\nu}\}) \quad , \tag{5.5}$$

$$\mathcal{L}_5 = \Lambda^{\alpha\beta\mu}{}_{\nu}({}^{\hat{\Gamma}}\tilde{K}_{\alpha\beta\mu}{}^{\nu} - \overset{0}{\nabla}_{\alpha}T_{\beta\mu}{}^{\nu}) \quad . \tag{5.6}$$

Before proceeding to the variation a few words are in order concerning the structure of this Lagrangian. We now denote an arbitrary generalized affine connection by $(\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu}, {}^{\hat{\Gamma}}K_{\mu\nu\lambda}{}^{\kappa})$. Moreover we use the linear connection $\Gamma_0 = (\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu})$ to define an origin $\hat{\Gamma}_0$ in the affine space of linear connections. Hence any other gauge $\hat{\Gamma}$ in the affine space can be expressed as $\hat{\Gamma} = \hat{\Gamma}_0 \oplus T$ so that each gauge relative to $\hat{\Gamma}_0$ is determined by a rank three tensor field $T_{\mu\nu}{}^{\lambda}$. The Christoffel symbols occuring in the constraint term $\mathcal{L}_4$ are defined in terms of the metric variable $g_{\mu\nu}$ while the curvature tensor $\overset{0}{R}_{\mu\nu\lambda\kappa}$ is the curvature tensor of the linear connection $\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu}$.

Referring to the transformation law (2.10) we see that the constraint term $\mathcal{L}_5$ introduces a pure gauge connection ${}^{\hat{\Gamma}}\tilde{K}_{\alpha\beta\mu}{}^{\nu}$ which vanishes identically in the $\hat{\Gamma}_0$ gauge. This pure gauge connection and its curvature are used in the terms $\mathcal{L}_3$ and $\mathcal{L}_2$, respectively, to insure translational invariance of the Lagrangian. The differences $({}^{\hat{\Gamma}}\Phi)^{\overset{*}{\mu}\overset{*}{\nu}\lambda} - ({}^{\hat{\Gamma}}\tilde{\Phi})^{\overset{*}{\mu}\overset{*}{\nu}\lambda}$ and ${}^{\hat{\Gamma}}K_{\alpha\beta\mu}{}^{\nu} - {}^{\hat{\Gamma}}\tilde{K}_{\alpha\beta\mu}{}^{\nu}$ occuring in (5.3) and (5.4) are differences between affine tensors and thus are translationally invariant tensors. The duals denoted by stars $*$ in (5.2) and (5.3) are Hodge duals determined by the metric tensor (see the Appendix).

Finally we consider the term $\mathcal{L}_1$. Substitution of the constraint given in (5.4) into $\mathcal{L}_1$ in the $\hat{\Gamma}_0$ gauge shows that $\mathcal{L}_1$ is the affine invariant way of writing the term $R_{\mu\nu\lambda\kappa}R^{\mu^*\nu\lambda^*\kappa}$ that plays such a central role in the Lanczos Lagrangian (1.5). In preparation for the variation we use the transformation law (2.10) to rewrite $\mathcal{L}_1$ in the general gauge $\hat{\Gamma}$ as

$$\mathcal{L}_1 = (\frac{1}{4})(\eta^{\alpha\beta\mu\nu}\eta^{\gamma\delta\sigma\rho})g_{\lambda\delta}g_{\rho\kappa}\Big((^{\hat{\Gamma}}K)_{\alpha\beta\gamma}{}^{\lambda}(^{\hat{\Gamma}}K)_{\mu\nu\sigma}{}^{\kappa} - (^{\hat{\Gamma}}K)_{\alpha\beta\gamma}{}^{\lambda}\overset{0}{\nabla}_{\mu}T_{\nu\sigma}{}^{\kappa} - (^{\hat{\Gamma}}K)_{\mu\nu\sigma}{}^{\kappa}\overset{0}{\nabla}_{\alpha}T_{\beta\gamma}{}^{\lambda} + \overset{0}{\nabla}_{\alpha}T_{\beta\gamma}{}^{\lambda}\overset{0}{\nabla}_{\mu}T_{\nu\sigma}{}^{\kappa}\Big) \quad . \tag{5.2b}$$

The total Lagrangian L is to be varied with respect to the basic variables $g_{\mu\nu}$, $\overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}$, $^{\hat{\Gamma}}K_{\mu\nu\lambda}{}^{\kappa}$ and $^{\hat{\Gamma}}\tilde{K}_{\mu\nu\lambda}{}^{\kappa}$. Moreover we recover the contraints by varying L with respect to the Lagrange multipliers. We note that the multiplier $\Lambda^{\mu\nu\lambda}{}_{\kappa}$ has no specific symmetries while the remaining multipliers have the index symmetries $B_{\nu\lambda\mu} = B_{[\nu\lambda]\mu}$, $\Sigma^{\mu\nu\lambda}{}_{\kappa} = [\Sigma^{\mu\nu\lambda}{}_{\kappa}]$, $P^{\mu\nu}{}_{\lambda} = P^{(\mu\nu)}{}_{\lambda}$. As the total variation of the Langrangian is rather complicated we consider the results of the individual variations sequentially.

Variation of L with respect to $P^{\mu\nu}{}_{\lambda}$ yields the constraint $\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$. Thus the linear component of the generalized affine connection $\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu}$ as well as the origin of the affine space of connections is the Levi-Civita connection of the metric. Variation of L with respect to $\Lambda^{\mu\nu\alpha}{}_{\beta}$ then implies the relation $^{\hat{\Gamma}}\tilde{K}_{\mu\nu\alpha}{}^{\beta} = \nabla_{\mu}T_{\nu\alpha}{}^{\beta}$, where ∇_{μ} now denotes covariant differentiation with respect to $\{^{\mu}_{\nu\lambda}\}$. This relation shows, as mentioned above, that $^{\hat{\Gamma}}\tilde{K}_{\mu\nu\alpha}{}^{\beta}$ is a flat affine connection that vanishes in the $\hat{\Gamma}_g$ gauge.

The variation of L with respect to $\Sigma^{\mu\nu\lambda}{}_{\kappa}$, when combined with the above results, yields

$$^{\hat{\Gamma}_g}K_{\mu\nu\lambda}{}^{\kappa} = R_{\mu\nu\lambda}{}^{\kappa} \quad , \tag{5.7}$$

where $R_{\mu\nu\lambda}{}^{\kappa}$ is now the Riemannian curvature tensor of the metric tensor. The above relations thus show that we have constrained the generalized affine connection to be the Riemannian affine connection $(\{^{\mu}_{\alpha\beta}\}, R_{\mu\nu\lambda}{}^{\kappa})$.

The final constraint follows from variation of L with respect to the Lagrange multiplier $B_{\mu\nu\lambda}$. This variation yields, upon using the previous relation,

$$^{\hat{\Gamma}_g}\Phi^{\overset{*}{\mu}\overset{*}{\nu}\lambda} = \nabla_{\mu}R^{\mu^*\nu\lambda^*\kappa} = 0 \quad , \tag{5.8}$$

which we recognize as the double-dual form of the full Riemannian Bianchi identities. Here it arises from the vanishing of the double-dual translational curvature.

The field equations which arise from variation with respect to the basic variables $g_{\mu\nu}$, $\overset{0}{\Gamma}{}^{\lambda}_{\mu\nu}$, $^{\hat{\Gamma}}\tilde{K}_{\alpha\beta\mu}{}^{\nu}$ and $^{\hat{\Gamma}}K_{\mu\nu\lambda}{}^{\kappa}$ are initially complicated, but they can be reduced to simplified form using (5.7)-(5.8). After omitting pure divergences that arise in the variation we obtain the following set of field equations:

$$\frac{\delta L}{\delta(^{\hat{\Gamma}}K_{\mu\nu\lambda}{}^{\kappa})} \Longrightarrow 0 = \Sigma^{\mu\nu\lambda}{}_{\kappa} + (\frac{1}{3})g_{\alpha\kappa}\eta^{\sigma\rho\mu\nu}\nabla_{\sigma}B^{\overset{*}{\lambda}\alpha}{}_{\rho} + 2g_{\alpha\kappa}R^{\overset{*}{\mu}\nu\overset{*}{\lambda}\alpha} \quad , \tag{5.9}$$

$$\frac{\delta L}{\delta({}^{\hat{\Gamma}}\tilde{K}_{\mu\nu\lambda}{}^{\kappa})} \Longrightarrow 0 = \Lambda^{\mu\nu\lambda}{}_{\kappa} - \Sigma^{\mu\nu\lambda}{}_{\kappa} - (\frac{1}{3}) g_{\alpha\kappa}\eta^{\sigma\rho\mu\nu}\nabla_{\sigma} B^{\overset{*}{\lambda}\alpha}{}_{\rho} \quad , \tag{5.10}$$

$$\frac{\delta L}{\delta g_{\lambda\kappa}} \Longrightarrow 0 = \big(2R_{\alpha\beta\gamma}{}^{(\lambda|} R^{\overset{*}{\alpha}\beta\overset{*}{\gamma}|\kappa)} - (\frac{1}{2}) g^{\lambda\kappa} R_{\alpha\beta\gamma\delta} R^{\overset{*}{\alpha}\beta\overset{*}{\gamma}\delta}\big)$$

$$+ (\frac{1}{2})\nabla_{\alpha}(P^{\alpha\kappa\lambda} + P^{\lambda\alpha\kappa} - P^{\kappa\lambda\alpha}) \quad , \tag{5.11}$$

$$\frac{\delta L}{\delta \overset{0}{\Gamma}{}^{\mu}_{\alpha\beta}} \Longrightarrow 0 = P^{\alpha\beta}{}_{\mu} + 2\nabla_{\sigma}\Sigma^{\sigma(\alpha\beta)}{}_{\mu}$$

$$+ (\frac{2}{3})(g_{\lambda\mu} B^{\overset{*}{\kappa}\lambda}{}_{\rho} R^{\overset{*}{\rho}(\alpha|}{}_{\kappa}{}^{|\beta)} - B^{(\overset{*}{\beta}|\lambda}{}_{\rho} R^{\overset{*}{\rho}|\alpha)}{}_{\mu\lambda}) \quad . \tag{5.12}$$

Lanczos showed that the Lagrange multipliers $P^{\mu\nu}{}_{\alpha}$ and $\rho^{\mu\nu}$ occurring in (1.5) could be eliminated in terms of the variables $H_{\mu\nu\alpha}$, $R^{\mu^*\nu\lambda^*\kappa}$, and $g_{\mu\nu}$. A similar result occurs in this affine reformulation. In particular, combining equations (5.9) and (5.12) we find

$$P^{\mu\nu}{}_{\lambda} = 0 \quad . \tag{5.13}$$

This result, namely the vanishing of the Lagrange multiplier constraining the connection to be the Levi-Civita connection of the metric, also occurs in the standard Palatini variational principle in general relativity. We note that (5.13) follows identically from (5.9) and (5.12) and does not depend on the choice of the tensor field $B^{\mu\nu}{}_{\lambda}$.

Substituting (5.13) into (5.11) yields

$$2R_{\alpha\beta\gamma}{}^{(\lambda|} R^{\overset{*}{\alpha}\beta\overset{*}{\gamma}|\kappa)} - (\frac{1}{2}) g^{\lambda\kappa} R_{\alpha\beta\gamma\delta} R^{\overset{*}{\alpha}\beta\overset{*}{\gamma}\delta} = 0 \quad . \tag{5.14}$$

This relation is the **Bach-Lanczos identity** [4,11] for a four-dimensional Riemannian geometry.

Equation (5.13) is the analog of the relation (2.14) in the 1962 Lanczos paper [3]. Indeed, in that work Lanczos obtained a rather complicated relation for $P^{\mu\nu}{}_{\lambda}$, and it is unclear whether or not that term can be reduced to zero using the other field equations found by Lanczos. Similarly equation (5.14) is the analog of the complicated relation (2.15) in Ref. [3], and it is not clear when the Lanczos equation (2.15) reduces to equation (5.14) above. On the other hand we note that Lanczos had derived (5.14) in 1938 [4] as an identity for four dimensional Riemannian geometries.

Finally we want to rederive the Lanczos potential structure equations (1.6) and (1.7). These will follow from the remaining two field equations (5.9) and (5.10) involving the Lagrange multipliers $\Sigma^{\mu\nu\lambda}{}_{\kappa}$, $\Lambda^{\mu\nu\lambda}{}_{\kappa}$ and $B^{\lambda\alpha}{}_{\rho}$. Adding (5.13) and (5.14) yields $-(\frac{1}{2})\Lambda^{\mu\nu\lambda}{}_{\kappa} = g_{\alpha\kappa} R^{\overset{*}{\mu}\nu\overset{*}{\lambda}\alpha}$ which, in view of (5.7), shows that $\Lambda^{\mu\nu\lambda}{}_{\kappa}$ is superfluous.

We noted above that (5.13) follows identically from (5.9) and (5.12) and is independent of the choice of the tensor field $B^{\mu\nu}{}_{\lambda}$. In fact all of the above results, and in particular the set of field equations (5.9)-(5.12), are consistent for any choice of $B^{\mu\nu}{}_{\lambda}$. Thus $B^{\mu\nu}{}_{\lambda}$ is completely arbitrary. This gauge freedom clearly reflects the arbitrariness in the choice of origin in the affine space of linear connections.

Thus let us consider $B^{\mu\nu}{}_{\lambda}$ in (5.9) to be some fixed, but arbitrary, tensor field. Next we solve (5.9) for $\Sigma^{\mu\nu\lambda}{}_{\kappa}$, take the double-dual of the result, and then form the bracket of the resulting equation. This yields

$$-(\frac{1}{2})[\Sigma_{\stackrel{*}{\mu\nu}\stackrel{*}{\kappa\lambda}}] = R_{\mu\nu\lambda\kappa} + (\frac{1}{3})[\nabla_\mu B_{\kappa\lambda\nu}] \quad . \tag{5.15}$$

Forming the conformal part of this equation using ${}_C[\quad]$ yields

$$-(\frac{1}{2})_C[\Sigma_{\stackrel{*}{\mu\nu}\stackrel{*}{\kappa\lambda}}] = C_{\mu\nu\kappa\lambda} + (\frac{1}{3})_C[\nabla_\mu B_{\kappa\lambda\nu}] \quad . \tag{5.16}$$

Using the Bampi-Caviglia integrability theorem [9] discussed in Section 4 we know that we can find a local solution $M_{\mu\nu\lambda}$ to the equation $(\frac{1}{8})_C[\Sigma_{\stackrel{*}{\mu\nu}\stackrel{*}{\kappa\lambda}}] = {}_C[\nabla_\mu M_{\kappa\lambda\nu}]$. Using this result back in (5.16) and rearranging terms we find the expression $C_{\mu\nu\kappa\lambda} = -4_C[\nabla_\mu H_{\kappa\lambda\nu}]$ for the conformal curvature tensor, where $H_{\kappa\lambda\nu} \equiv M_{\kappa\lambda\nu} + (\frac{1}{12})B_{\kappa\lambda\nu}$. Thus we have recovered the Lanczos potential structure equation (1.7).

Next we use (5.7) to rewrite $C_{\mu\nu\kappa\lambda} = -4_C[\nabla_\mu H_{\kappa\lambda\nu}]$ in the form ${}_C[{}^{\hat{\Gamma}_g}K_{\mu\nu\kappa\lambda}] = -4_C[\nabla_\mu H_{\kappa\lambda\nu}]$. As described in Section 4 this last expression is equivalent to the expression ${}_C[{}^{\hat{\Gamma}_g \oplus 4H}K_{\mu\nu\kappa\lambda}] = 0$. The final structure equation (1.6) now follows from Theorem 4.2.

6. Conclusions

A central idea in modern physics is to use gauge fields to model the fundamental forces of nature, and the geometrical setting for gauge theories is that of connections on principal fiber bundles. In this paper we have shown that the Lanczos H-tensor formulation of Riemannian geometrical structure can be modeled in a natural way using a generalized affine connection on a particular affine frame bundle $A^1_2 M$.

It is well-known that the set of all linear connections on a manifold M has a natural affine structure, and in Section 2 we introduced the basic machinery needed to treat linear connections as affine tensors of type (1,2). In particular we argued that the basic geometrical object needed for such affine tensors is a generalized affine connection on $A^1_2 M$. Each such affine connection may be specified by a pair $(\Gamma^\mu_{\nu\kappa}, {}^{\hat{\Gamma}_0}K_{\mu\nu\kappa}{}^\lambda)$ where $\Gamma^\mu_{\nu\kappa}$ denotes a linear connection and ${}^{\hat{\Gamma}_0}K_{\mu\nu\kappa}{}^\lambda$ is a type (1,3) tensor field representing the translational component of the affine connection with respect to the origin $\hat{\Gamma}_0$ in the affine space of linear connections. We noted that an affine connection may be specified by a linear connection $\Gamma^\mu_{\nu\kappa}$ and its curvature tensor $R_{\mu\nu\kappa}{}^\lambda$, namely as the pair $(\Gamma^\mu_{\nu\kappa}, R_{\mu\nu\kappa}{}^\lambda)$. We were then led to the formula (cf. equation (2.13))

$$D_\mu(\hat{\Gamma}_1)^\nu_{\lambda\kappa} = \nabla_\mu(T_{\nu\lambda}{}^\mu) + R_{\mu\lambda\kappa}{}^\nu \tag{6.1}$$

for the affine covariant derivative of the linear connection $(\Gamma_1)^\nu_{\lambda\kappa}$ when it is treated as an affine tensor. This fundamental relation plays a key role in the later developments.

In order to make contact with the Lanczos formulation of Riemannian geometrical structure we considered a four-dimensional spacetime (M, g), and we chose the origin of the affine space of linear connections to be $\Gamma_g \longrightarrow \hat{\Gamma}_g$, where Γ_g denotes the Riemannian connection coefficients constructed from the given metric tensor g. Furthermore, given Γ_g

we defined a unique generalized affine connection, or **Riemannian affine connection**, by choosing the pair $(\{^{\mu}_{\nu\lambda}\}, R_{\mu\nu\kappa}{}^{\lambda})$ on M.

In Sections 3 and 4 we considered $A^1_2(4)$ induced translations of the form $\hat{\Gamma}' \longrightarrow \hat{\Gamma}' \oplus A$, where A is a type (1,2) tensor which satisfies the symmetry property $A_{\mu\nu}{}^{\alpha} = -A_{\nu\mu}{}^{\alpha}$. We then showed that the entire Lanczos H-tensor formalism for Riemannian geometrical structure arises from a **gauge-fixing condition** on the translational connection ${}^{\hat{\Gamma}'}K_{\mu\nu\alpha\beta}$, namely the condition

$$ {}_{C}[{}^{\hat{\Gamma}_g \oplus \tilde{A}}K_{\mu\nu\alpha\beta}] = 0 \quad . \tag{6.2}$$

Without loss of generality we have replaced the tensor $A_{\alpha\beta\nu}$ with the tensor $\tilde{A}_{\alpha\beta\nu}$ which satisfies the additional symmetry condition $\tilde{A}_{[\alpha\beta\nu]} = 0$. We showed in Section 4 that this single condition reproduces the Lanczos relations (1.6) and (1.7) when we make the identifications

$$ {}^{\hat{\Gamma}'}Q_{\mu\nu} = (\frac{1}{2})\left([{}^{\hat{\Gamma}'}K_{\mu\nu}] - (\frac{1}{4})g_{\mu\nu}[{}^{\hat{\Gamma}'}K]\right) \tag{6.3a}$$

$$ {}^{\hat{\Gamma}'}q = -(\frac{1}{24})[{}^{\hat{\Gamma}'}K] \tag{6.3b}$$

$$ H_{\mu\nu\kappa} = (\frac{1}{4})\tilde{A}_{\mu\nu\kappa} \quad . \tag{6.3c}$$

Note that the geometrical objects defined in (6.3a) and (6.3b) are constructed directly from the translational component ${}^{\hat{\Gamma}'}K_{\mu\nu\kappa}{}^{\lambda}$ of the generalized affine connection and are thus affine tensors, transforming as in (3.5a) and (3.5b) under translations. In particular those transformation formulas involve the derivatives of the translation tensor $A_{\mu\nu\kappa}$. On the otherhand the transformation law (1.9a) for the Lanczos $H_{\mu\nu\kappa}$ involves the transformation vector itself rather than its derivatives. In fact $H_{\mu\nu\kappa}$ defined in (6.3c) is a tensor that fixes the gauge in which (6.2) is satisfied. We call this gauge the **Lanczos gauge**. It is unique up to translations of the form $\tilde{A}_{\mu}{}^{\nu}{}_{\kappa} = 4(V_{\mu}\delta^{\nu}_{\kappa} - V^{\nu}g_{\mu\kappa})$ where V^{μ} is an arbitrary vector. Hence the Lanczos $H_{\mu\nu\lambda}$ is not uniquely defined and is thus also an affine tensor. When the translations in $A^1_2(4)$ are restricted to the Lanczos symmetry subgroup, namely to transformations induced by (1,2) tensors of the form given above, then the transformations of the affine tensor ${}^{\hat{\Gamma}'}Q_{\mu\nu}$ and the affine scalar ${}^{\hat{\Gamma}'}q$ given in (3.5a) and (3.5b) reduce to the special forms (1.9b) and (1.9c) considered by Lanczos.

We point out that in the present developments we have used only a small part of the available geometrical structure associated with generalized affine connections for type (1,2) affine tensors. Specifically, the entire Lanczos H-tensor formalism is associated with the single gauge-fixing condition (6.2) applied to the specific generalized Riemannian affine connection $(\Gamma_g, {}^{\hat{\Gamma}_g}\phi)$. There was no apparent role for the generalized affine curvature $(\Omega_g, {}^{\hat{\Gamma}_g}\Phi)$ to play in the new affine formulation. However, we showed in Section 5 that the translational affine curvature can be used to reconstruct the Lanczos Lagrangian (1.5) in terms of affine invariants. The essential field equations resulting from the affine variational principle were the Riemannian Bianchi and Bach-Lanczos identities. The "Riemannian Bianchi Identity" constraint $H_{\mu\nu\kappa}\nabla_{\lambda}R^{\overset{*}{\mu}\nu\overset{*}{\kappa}\lambda}$ occuring in (1.5) was reexpressed in Section 5 in terms of the translational curvature as $H_{\mu\nu\kappa}({}^{\hat{\Gamma}_g}\Phi^{\overset{*}{\mu}\overset{*}{\nu}\kappa})$ where $\Phi^{\overset{*}{\mu}\overset{*}{\nu}\kappa}$ is a certain double-dual

(2.15) of the translational curvature. This seems a rather minimal role for the geometrical object that plays such a prominent role in gauge theories.

In order to see what more significant role might be available for the translational curvature we consider a unified theory of gravitation and electromagnetism which has been proposed recently[8]. The geometrical arena for this theory is another affine frame bundle A_1M over a four-dimensional spacetime manifold (M, g). The structure group of A_1M is the Poincaré group $P(4) = O(1,3) \circledS \mathbb{R}^{4*}$ and A_1M is a trivial $\mathbb{R}^{4*}$ principal bundle over the linear orthonormal frame bundle OM. In particular, a point of A_1M is a triple $(p, e_\mu, \hat{t})$ where $p \in M$, (e_μ) is an orthonormal linear frame at p, and $\hat{t}$ is an **affine cotangent vector** at p. Thus a cotangent **vector** s induces a translation of the form $\hat{t} \longrightarrow \hat{t} \oplus s$ that is the analogue of the translations $\hat{\Gamma} \longrightarrow \hat{\Gamma} \oplus A$ used above for linear connections thought of as affine tensors.

A generalized affine connection on A_1M corresponds to a unique pair $(\{^{\mu}_{\nu\kappa}\}, {}^{\hat{t}}K_{\mu\nu})$ on M, and conversely. The associated curvature of the generalized affine connection can thus be expressed as the pair $(R_{\mu\nu\kappa}{}^{\lambda}, {}^{\hat{t}}\Phi_{\mu\nu\kappa})$ on M, and the translational component of the curvature can be written in terms of the connection as

$$ {}^{\hat{t}}\Phi_{\mu\nu\kappa} = \nabla_\mu({}^{\hat{t}}K_{\kappa\nu}) - \nabla_\nu({}^{\hat{t}}K_{\kappa\mu}) \tag{6.4} $$

in any translational gauge $\hat{t}$. In the P(4) theory the Maxwell field strength $F_{\mu\nu}$ is identified with the translational component of the affine connection in a certain gauge $\hat{0}$ that is defined physically in terms of the observational meaning of the Lorentz force law. In this $\hat{0}$ gauge the translational component ${}^{\hat{0}}K_{\mu\nu}$ of the affine connection is defined as ${}^{\hat{0}}K_{(\mu\nu)} = 0$ and ${}^{\hat{0}}K_{[\mu\nu]} = -F_{\mu\nu}$. It follows [12] that with this identification an affine geodesic equation is equivalent to the Lorentz force law. Furthermore, the source-free Maxwell equations

$$ (a) \quad \nabla_\mu F_\nu{}^{\mu} = 0 \quad , \quad (b) \quad \nabla_{[\mu}F_{\nu\lambda]} = 0 \quad , \tag{6.5} $$

can be geometrized completely in terms of the components of the translational part of the generalized affine curvature ${}^{\hat{0}}\Phi_{\mu\nu\kappa}$. Indeed, it follows directly from (6.4) and ${}^{\hat{0}}K_{\mu\nu} = -F_{\mu\nu}$ that (6.5a) and (6.5b) can be rewritten as

$$ (a) \quad {}^{\hat{0}}\Phi_{\mu\nu}{}^{\mu} = 0 \quad , \quad (b) \quad {}^{\hat{0}}\Phi_{[\mu\nu\kappa]} = 0 \tag{6.6} $$

Let us now return to the affine geometry of linear connections discussed in this paper. We recall that in reformulating the Lanczos H-tensor structure for Riemannian geometries we chose the Riemannian connection Γ_g to be the origin $\Gamma_g \longrightarrow \hat{\Gamma}_g$ in the affine space of linear connections, and relative to this choice of origin we chose the special Riemannian affine connection $(\{^{\mu}_{\nu\lambda}\}, {}^{\hat{\Gamma}_g}K_{\mu\nu\kappa}{}^{\lambda})$ where ${}^{\hat{\Gamma}_g}K_{\mu\nu\kappa}{}^{\lambda} = R_{\mu\nu\kappa}{}^{\lambda}$. Hence the associated affine curvature was given by the pair $(R_{\mu\nu\kappa}{}^{\lambda}, {}^{\hat{\Gamma}_g}\Phi_{\mu\nu\kappa\lambda}{}^{\rho})$ where (see (2.14))

$$ {}^{\hat{\Gamma}_g}\Phi_{\mu\nu\kappa\lambda}{}^{\rho} = 2\nabla_{[\mu}R_{\nu]\kappa\lambda}{}^{\rho} \quad . \tag{6.7} $$

If we pursue the obvious parallel between the structure of this geometry and the geometry of the P(4) theory described above we are led to consider the analogues of the $\mathbb{R}^{4*}$ field equations (6.6a) and (6.6b), namely

$$ (a) \quad {}^{\hat{\Gamma}_g}\Phi_{\mu\nu}{}^{\mu}{}_{\lambda}{}^{\rho} = 0 \quad , \quad (b) \quad {}^{\hat{\Gamma}_g}\Phi_{[\mu\nu\kappa]\lambda}{}^{\rho} = 0 \quad . \tag{6.8} $$

respectively. It follows directly from (6.7) that equations (6.8) are equivalent to

$$(a) \quad \nabla_\mu R_{\lambda\cdot\nu}^{\;\rho\;\;\mu} = 0 \quad , \quad (b) \quad \nabla_{[\mu} R_{\nu\kappa]\lambda}{}^{\rho} = 0 \quad . \tag{6.9}$$

Clearly (6.9b) is the full uncontracted Riemannian Bianchi Identity. The equations (6.9a) and (6.9b) taken together are precisely the source-free field equations of Yang's gravitational gauge theory [13] of the non-integrable phase factor. We have shown that these equations can be geometrized in a natural way in terms of the translational component ${}^{\hat{\Gamma}_g}\Phi_{\mu\nu\kappa\lambda}{}^{\rho}$ of the affine curvature. Moreover, these source-free gauge field equations arise in a manner which is completely analogous to the way the Maxwell equations arise in the P(4) theory.

We remarked in Section 5 that the field equations derived from the affine variational principle based on (5.1)-(5.6) do not determine the origin $\hat{\Gamma}_0$ in the affine space of linear connections. In the variational principle we chose $\hat{\Gamma}_0 = \hat{\Gamma}_g$ on the basis of simplicity so as to not introduce any additional unknown structure into the theory, but the affine Lagrangian (5.1)-(5.6) does not determine the "origin field = metric connection" but led instead to Riemannian curvature identities. This is in harmony with the Lanczos method of studying the structure of all 4 dimensional Riemannian geometries, since the field equations derived from the Lanczos Lagrangian (1.5) also leave the metric connection undetermined. One can only determine the metric from a definite set of field equations. In this affine setting Yang's source-free field equations represent equations determining the origin $\hat{\Gamma}_g$ in the affine space of connections. Thus although numerous objections have been raised[9] to Yang's theory as a theory of gravitation, our results above indicate that perhaps some version of the theory may yet be useful in a generalized affine theory of linear connections.

APPENDIX

In this appendix we introduce certain notations and conventions which are used throughout this paper. Let M denote a four-dimensional spacetime manifold with metric tensor $g = (g_{\mu\nu})$ of signature -2. Throughout this paper we follow the conventions of Synge [20] relating to Hodge duals. In particular, the dual $\overset{*}{F}{}^{\mu\nu}$ of an antisymmetric tensor $F_{\mu\nu}$ is defined by $\overset{*}{F}{}^{\mu\nu} = (\frac{1}{2})\eta^{\mu\nu\alpha\beta}F_{\alpha\beta}$. Here $\eta^{\mu\nu\alpha\beta}$ is the permutation tensor based on the metric tensor g defined so that it satisfies $\eta^{\mu\nu\alpha\beta}\eta_{\mu\nu\sigma\rho} = -2\left(\delta^\alpha_\sigma\delta^\beta_\rho - \delta^\alpha_\rho\delta^\beta_\sigma\right)$.

In 1962 Lanczos [3] showed that the conformal tensor could be written as

$$\begin{aligned} C_{\mu\nu\alpha\beta} = {} & (\nabla_\beta H_{\mu\nu\alpha} - \nabla_\alpha H_{\mu\nu\beta} + \nabla_\nu H_{\alpha\beta\mu} - \nabla_\mu H_{\alpha\beta\nu}) \\ & - (\frac{1}{2})\left(g_{\mu\alpha}(H_{\nu\beta} + H_{\beta\nu}) - g_{\mu\beta}(H_{\nu\alpha} + H_{\alpha\nu})\right) \\ & - (\frac{1}{2})\left(g_{\nu\beta}(H_{\mu\alpha} + H_{\alpha\mu}) - g_{\nu\alpha}(H_{\mu\beta} + H_{\beta\mu})\right) \\ & + (\frac{1}{3})(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})H^\sigma_\sigma \end{aligned} \tag{A1}$$

where $H_{\mu\nu\alpha} = -H_{\nu\mu\alpha}$ is the Lanczos tensor and $H_{\nu\beta} = \nabla_\alpha H_{\nu\cdot\beta}^{\;\alpha} - \nabla_\beta H_{\nu\alpha}{}^{\alpha}$. The H-tensor that appears in (A1) is assumed to also satisfy the algebraic condition $H_{[\mu\nu\alpha]} = 0$.

In this paper we deal with rank-four tensors which, initially, do not necessarily have any specific index symmetries. It is convenient to have an operation that converts a given tensor $B_{\mu\nu\alpha\beta}$ into a new tensor with the index symmetries of the curvature tensor. Following Lanczos [3] we define a "bracket operation" on rank four tensors as follows:

$$[B_{\mu\nu\alpha\beta}] = (\frac{1}{8})\Big(B_{\mu\nu\alpha\beta} - B_{\nu\mu\alpha\beta} + B_{\alpha\beta\mu\nu} - B_{\alpha\beta\nu\mu} + B_{\nu\mu\beta\alpha} - B_{\mu\nu\beta\alpha} + B_{\beta\alpha\nu\mu} - B_{\beta\alpha\mu\nu} + (\frac{1}{3})\eta_{\mu\nu\alpha\beta}(\eta^{\theta\phi\sigma\rho}B_{\theta\phi\sigma\rho})\Big) \tag{A2}$$

More details about this operator can be found in the paper by Bampi and Caviglia [9]. We note that it follows from (A2) that the new "bracketed object" $[B]$ has the index symmetries of the Riemannian curvature tensor and satisfies $[B]_{[\mu\nu\alpha]\beta} = 0$. The numerical factors used in (A2) guarantee that if $B_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}$, then $[B_{\mu\nu\alpha\beta}] = R_{\mu\nu\alpha\beta}$.

In order to parallel the decomposition given in (1.3), we split the bracket operation defined in (A2) into two pieces, namely $[B_{\mu\nu\alpha\beta}] = {}_C[B_{\mu\nu\alpha\beta}] + {}_L[B_{\mu\nu\alpha\beta}]$ where ${}_C[\ \ ]$ is a conformal, or trace-free part, and ${}_L[\ \ ]$ denotes a local, or trace part. Specifically, these quantities are defined by

$${}_C[B_{\mu\nu\alpha\beta}] = [B_{\mu\nu\alpha\beta}] + 2[g_{\mu\alpha}[B_{\nu\beta}]] - (\frac{1}{3})[g_{\mu\alpha}g_{\nu\beta}[B]] \tag{A3}$$

$${}_L[B_{\mu\nu\alpha\beta}] = -2[g_{\mu\alpha}[B_{\nu\beta}]] + (\frac{1}{3})[g_{\mu\alpha}g_{\nu\beta}[B]] \tag{A4}$$

In these relations we have used the definitions $[B_{\nu\alpha}] = g^{\mu\beta}[B_{\mu\nu\alpha\beta}]$ and $[B] = g^{\nu\alpha}[B_{\nu\alpha}]$. ${}_C[B_{\mu\nu\alpha\beta}]$ has the same index symmetries as the conformal curvature tensor.

Next we consider a rank three tensor $A_{\mu\nu\alpha}$ that satisfies $A_{\mu\nu\alpha} = -A_{\nu\mu\alpha}$. Following Bampi and Caviglia [9] we define a related tensor $\tilde{A}_{\mu\nu\alpha}$ by

$$\tilde{A}_{\mu\nu\alpha} = A_{\mu\nu\alpha} + (\frac{1}{6})\eta_{\mu\nu\alpha\beta}(\eta^{\gamma\delta\sigma\beta}A_{\gamma\delta\sigma}) \tag{A5}$$

This associated tensor has the additional index symmetry $\tilde{A}_{[\mu\nu\alpha]} = 0$. It is straight forward to show using (A2) that

$$[\nabla_\beta A_{\mu\nu\alpha}] = [\nabla_\beta \tilde{A}_{\mu\nu\alpha}] \tag{A6}$$

Moreover, calculating ${}_C[\nabla_\beta(4H_{\mu\nu\alpha})]$ using (A3) we may rewrite (A1) in the forms

$$\begin{aligned} C_{\mu\nu\alpha\beta} &= 4_C[\nabla_\beta H_{\mu\nu\alpha}] \\ &= -4_C[\nabla_\mu H_{\alpha\beta\nu}] \end{aligned} \tag{A7}$$

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FOOTNOTES

1. Department of Physics, Box 8202, North Carolina State University, Raleigh, NC 27695-8202.

2. Department of Mathematics, Box 8205, North Carolina State University, Raleigh, NC 27695-8205.

3. For a discussion of Pontrjagin classes see, for example, Nash and Sen [5].

4. Lanczos [3] showed that the extra degrees of freedom associated with $H_{\mu\nu\lambda}$ could be eliminated by imposing the gauge conditions $H_{[\mu\nu\lambda]} = 0$, ${H_{\mu\nu}}^{\mu} = 0$, and $\nabla_\lambda {H_{\mu\nu}}^{\lambda} = 0$. Neverthess, for the sake of generality we shall assume throughout this paper that $H_{\mu\nu\lambda}$ satisfies only the first of these three conditions.

5. Exact solutions for the Lanczos tensor within the context of Einstein's general theory of relativity have been derived by Takeno [6] and Novello and Velloso [7]. There has been a large amount of more recent interest in the Lanczos formalism and the reader is referred to the papers by Atkins and Davis [8], Bampi and Caviglia [9], Roberts [10] and references therein.

6. See, for example, Dodson and Posten [14] for an introduction to affine spaces.

7. An affine variational principle which yields the coupled Einstein-Maxwell affine field equations associated with the P(4) theory of gravity and electromagnetism has been developed by Chilton and Norris [16].

8. The P(4) theory was originally introduced by Norris [12] in 1985. Further developments can be found in the works by Kheyfets and Norris [17], Chilton and Norris [16] and Norris [18].

9. See, for example, the discussion by Fairchild [19].

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