GEOMETRIC STRUCTURES IN FIELD THEORY

Manuel de León

IMFF (C.S.I.C.), Serrano 123, 28006, Madrid, Spain

E-Mail: mdeleon@fresno.csic.es

Michael McLean and Larry K. Norris

Department of Mathematics, North Carolina State Univerisity

Box 8205, Raleigh, North Carolina, USA

E-Mail: mamclean@unity.ncsu.edu, lkn@math.ncsu.edu

Angel Rey Roca and Modesto Salgado

Departamento de Xeometría e Topoloxía, University of Santiago de Compostela

Facultad de Matematicas, 15706 Santiago de Compostela, Spain

E-Mail: modesto@zmat.usc.es

Abstract

This review paper is concerned with the generalizations to field theory of the tangent and cotangent structures and bundles that play fundamental roles in the Lagrangian and Hamiltonian formulations of classical mechanics. The paper reviews, compares and constrasts the various generalizations in order to bring some unity to the field of study. The generalizations seem to fall into two categories. In one direction some have generalized the geometric structures of the bundles, arriving at the various axiomatic systems such as k-symplectic and k-tangent structures. The other direction was to fundamentally extend the bundles themselves and to then explore the natural geometry of the extensions. This latter direction gives us the multisymplectic geometry on jet and cojet bundles and n-symplectic geometry on frame bundles.

Contents

1 Introduction

This review paper is inspired by the geometric formulations of the Lagrangian and Hamiltonian descriptions of classical mechanics. The mathematical arenas of these well-known formulations are respectively, the tangent and cotangent bundles of the configuration space. Over the years many have sought to study classical field theory in analogous ways, using various generalizations or extensions of the tangent and cotangent bundles and/or their structures. No one has yet achieved a perfect formalism, but there are beautiful and useful results in many arenas.

The generalizations seem to fall into two categories. In one direction, some have generalized the geometric structures of these bundles, sometimes arriving at a formalism pertinent to field theory. Another direction was to fundamentally extend the bundles themselves and then explore the natural geometry of the extensions. The former gives us the various axiomatic systems such as k-symplectic and k-tangent structures. The latter gives us the multisymplectic geometry on jet and cojet bundles and n-symplectic geometry on frame bundles.

1.1 Cotangent-like structures

The first step in this direction of generalization was the development of symplectic geometry [1]. Later, around 1960, Bruckheimer [2] introduced the notion of almost cotangent structures. These were futher investigated by Clark and Goel [3] in 1974. In both cases the canonical 2-form became the model from which axioms were designed.

Between 1987 and 1991, several independent and closely related generalizations were developed. Polysymplectic geometry [4], almost k-cotangent structures [5, 6], and k-symplectic geometry [7, 8] were based around the natural structure of the k-cotangent bundle. This bundle, which can be thought of as the fiberwise product of the cotangent bundle k times, has a k-tuple of 1-forms with which one works. Also, the development of the n-symplectic geometry of the frame bundle and its $\mathbb{R}^n$-valued soldering 1-form θ began during this time period [9, 10, 11, 12]. While the development of k-tangent structures and k-symplectic geometry had purely geometric motivations, polysymplectic geometry was created to study field theory and m-symplectic geometry sought to generalize Hamiltonian mechanics.

1.2 Tangent-like structures

Around 1960, the theory of almost tangent structures was developed by Clark and Bruckheimer [13] and Eliopoulos [14] separately. Almost tangent structures are generalizations of the tangent bundle. The canonical vector valued one-form J, viewed as the object of central interest, was axiomatized.

Almost k-tangent structures [15, 16] arose around 1988 as a generalization of the geometry of the k-tangent bundle. This bundle is, among other interpretations, the fiberwise product of the tangent bundle with itself k times. A section of this bundle is equivalent to a k-tuple of vector fields. The central geometric object becomes a k-tuple of $J's$.

Another version of the tangent structure arises on the jet bundle (see [17]). This is a very broad level of generalization since the idea of the jet of a section generalizes and incorporates the notions of tangent vectors, cotangent vectors, k-tangent vectors, and k-cotangent vectors. Such geometry has clear importance to field theory since one can envision any type of field as a section of a fiber bundle.

We present here also a new tangent-like structure, namely a canonically defined set of tensor fields J^i, $i = 1...n$ on the bundle of frames LM of a manifold M. This tangent-like structure will be shown to induce the tangent structure on TM.

1.3 Interconnections, and plan of the paper

In this review paper our goal is to identify and clarify important connections between the various structures mentioned above. We also will consider relationships between some of the formalisms built on top of these structures.

The k-cotangent, k-symplectic, and polysymplectic structures are nested generalizations with k-cotangent being the most specific. The n-symplectic geometry of the frame bundle is also an example of a polysymplectic structure. Later in the paper, we will draw some interesting connections between the frame bundle and the k-cotangent bundle.

The frame bundle is an interesting case since in addition to having a cotangent-like structure, it also has a tangent-like structure. Exploiting the natural correlation of frames and co-frames, we can define an n-tuple of Js in addition to the m-tuple of θs mentioned earlier. These objects acquire additional properties and relationships on the frame bundle.

The vector valued one-form S_α on the jet bundle is later shown to be directly related to the other tangent structures in the special cases where they are comparable. Additionally, using new results regarding the adapted frame bundle we show a similar relationship between the k-tangent structure there and the S_α on the jet bundle.

Venturing into the realm of multi-symplectic geometry, we show how the canonical multi-symplectic form on the cojet bundle is tied to the canonical k-symplectic structures we discuss. Moreover we show how the Cartan-Hamilton-Poincaré n-form on $J^1\pi$ is induced from the m-symplectic structure on $L_\pi E$.

What we strive to do in this paper is to unify perspectives. We show similarities and differences among the approaches and draw strong correlations. Since no one geometry has emerged as dominant, it is important that everyone be aware of the options. We hope this work may serve as a guidebook and translation table for those desiring to explore other formalisms.

All the manifolds are supposed to be smooth. The differential of a mapping $F : M \longrightarrow N$ at a point $x \in M$ will be denoted by $F_*(x)$ or $TF(x)$. The induced tangent mapping will be denoted as $TF : TM \longrightarrow TN$.

The names of the various theories are different, yet two names are so similar that we feel it necessary to introduce the following convention that will be followed throughout the paper.

- We use the term *k-symplectic geometry* to refer to the works of Awane and the works of de León, Salgado, et. al.

- We use the terms *n-symplectic geometry* and/or *m-symplectic geometry* to refer to the works of Norris et. al.

2 Spaces with tangent-like structures

In this section we first recall the definitions and main properties of almost tangent and almost k-tangent structures. We describe the canonical n-tangent structure of the frame bundle LM of an n-dimensional manifold M in terms of the soldering form.

Secondly we recall Saunders's construction of the vector valued 1-form S_α. This 1-form is a generalization to field theories defined on jet bundles of fibered manifolds, of the almost tangent structure.

2.1 Almost tangent structures and TM

An almost tangent structure J on a $2n$-dimensional manifold M is tensor field of type $(1,1)$ of constant rank n such that $J^2 = 0$. The manifold M is then called an almost tangent manifold. Almost tangent structures were introduced by Clark and Bruckheimer [13] and Eliopoulos [14] around 1960 and have been studied by numerous authors (see [18, 19, 20, 21, 22, 23, 24, 25, 26]).

The canonical model of these structures is the tangent bundle $\tau_M : TM \to M$ of an arbitrary manifold M. Recall that for a vector X_x at a point $x \in M$ its vertical lift is the vector on TM given by

$$X_x^V(v_x) = \frac{d}{dt}(v_x + tX_x)_{|t=0} \in T_{v_x}(TM)$$

for all points $v_x \in TM$.

The canonical tangent structure J on TM is defined by

$$J_{v_x}(Z_{v_x}) = ((\tau_M)_*(v_x)Z_{v_x})^V_{v_x}$$

for all vectors $Z_{v_x} \in T_{v_x}(TM)$, and it is locally given by

$$J = \frac{\partial}{\partial v^i} \otimes dx^i \tag{1}$$

with respect the bundle coordinates on TM. This tensor J can be regarded as the vertical lift of the identity tensor on M to TM [27].

The integrability of these structures, which means the existence of local coordinates such that the tensor field J is locally given like as in (1), is characterized as follows.

Proposition 2.1 *An almost tangent structure J on M is integrable if and only if the Nijenhuis tensor N_J of J vanishes.* ■

Crampin and Thompson [20] proved that an integrable almost tangent manifold M satisfying some natural global hypotheses is essentially the tangent bundle of some differentiable manifold.

2.2 Almost k-tangent structures and T^1_kM

The almost k-tangent structures were introduced as generalization of the almost tangent structures [15, 16].

Definition 2.2 *An* almost k-tangent structure J *on a manifold M of dimension $n+kn$ is a family $(J^1,\dots,J^k)$ of tensor fields of type $(1,1)$ such that*

$$J^A\circ J^B=J^B\circ J^A=0,\qquad rank\, J^A=n,\qquad Im\, J^A\cap(\oplus_{B\neq A}Im\, J^B)=0, \tag{2}$$

for $1\leq A,B\leq k$. In this case the manifold M is then called an almost k-tangent manifold.

The canonical model of these structures is the k-tangent vector bundle $T^1_kM=J^1_0(\mathbf{R}^k,M)$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with source at $0\in\mathbf{R}^k$ and with projection map $\tau(j^1_0\sigma)=\sigma(0)$. This bundle is also known as the *tangent bundle of k^1-velocities* of M [27].

The manifold T^1_kM can be canonically identified with the Whitney sum of k copies of TM, that is

$$\begin{array}{rcl} T^1_kM & \equiv & TM\oplus\cdots\oplus TM,\\ j^1_0\sigma & \equiv & (j^1_0\sigma_1=v_1,\dots,j^1_0\sigma_k=v_k) \end{array}$$

where $\sigma_A=\sigma(0,\dots,t,\dots,0)$ with $t\in\mathbf{R}$ at position A and $v_A=(\sigma_A)_*(0)(\frac{d}{dt}\big|_0)$.

If (x^i) are local coordinates on $U\subseteq M$ then the induced local coordinates (x^i,v^i_A), $1\leq i\leq n$, $1\leq A\leq k$, on $\tau^{-1}(U)\equiv T^1_kU$ are given by

$$x^i(j^1_0\sigma)=x^i(\sigma(0)),\qquad v^i_A(j^1_0\sigma)=\frac{d}{dt}(x^i\circ\sigma_A)|_{t=0}=v_A(x^i)\,.$$

Definition 2.3 *For a vector X_x at M we define its* vertical A-lift $(X_x)^A$ *as the vector on $T^1_k M$ given by*

$$(X_x)^A(j^1_0\sigma) = \frac{d}{dt}((v_1)_x, \ldots, (v_{A-1})_x, (v_A)_x + tX_x, (v_{A+1})_x \ldots, (v_k)_x)|_{t=0} \in T_{j^1_0\sigma}(T^1_k M)$$

for all points $j^1_0\sigma \equiv ((v_1)_x, \ldots, (v_k)_x)) \in T^1_k M$.

In local coordinates we have

$$(X_x)^A = \sum_{i=1}^{n} a^i \frac{\partial}{\partial v^i_A} \tag{3}$$

for a vector $X_x = a^i \, \partial/\partial x^i$.

The *canonical vertical vector fields* on $T^1_k M$ are defined by

$$C^A_B(x, X_1, X_2, \ldots, X_k) = (X_B)^A \tag{4}$$

and are locally given by $C^A_B = v^i_B \frac{\partial}{\partial v^i_A}$. The *canonical k-tangent structure* $(J^1, \ldots, J^k)$ on $T^1_k M$ is defined by

$$J^A(Z_{j^1_0\sigma}) = (\tau_*(Z_{j^1_0\sigma}))^A$$

for all vectors $Z_{j^1_0\sigma} \in T_{j^1_0\sigma}(T^1_k M)$. In local coordinates we have

$$J^A = \frac{\partial}{\partial v^i_A} \otimes dx^i \tag{5}$$

The tensors J^A can be regarded as the $(0, \ldots, 1_A, \ldots, 0)$-lift of the identity tensor on M to $T^1_k M$ defined in [27].

We remark that an almost 1-tangent structure is an almost tangent structure.

In [15, 16] the almost k–tangent structures are described as G-structures, and the integrability of these structures, which is defined as the existence of local coordinates such that the tensor fields J^A are locally given as in (5), is characterized by following proposition.

Proposition 2.4 *An almost k-tangent structure $(J^1, \ldots, J^k)$ on M is integrable if and only if $\{J^A, J^B\} = 0$ for all $1 \leq A, B \leq k$, where*

$$\{J^A, J^B\}(X, Y) = [J^A X, J^B Y] + J^A J^B [X, Y] - J^A[X, J^B Y] - J^B[J^A X, Y]\,,$$

for any vector fields X, Y on M. ■

In [15, 16] it is proved (in a way analogous to [20]) that an integrable almost tangent manifold M satisfying some natural global hypotheses is essentially the k-tangent bundle of some differentiable manifold.

2.3 The canonical n-tangent structure of LM

We shall show that LM has an intrinsic n-tangent structure described in terms of the soldering form and fundamental vertical vectors fields.

Let M be a n-dimensional manifold and $\lambda_M : LM \to M$ the principal fiber bundle of linear frames of M. A point u of LM will be denoted by the pair (x, e_i) where $x \in M$ and $(e_1, e_2, \ldots, e_n)_x$ denotes a linear frame at x. The projection map $\lambda_M : LM \to M$ is defined by $\lambda_M(x, e_i) = x$.

If (U, x^i) is a chart on M then we can introduce two different coordinates on $\lambda_M^{-1}(U)$. First consider the *coframe* or *n-symplectic momentum* coordinates (x^i, π^i_j) on $\lambda_M^{-1}(U)$ defined by

$$x^i(u) = x^i(x) \ , \qquad \pi^i_j(u) = e^i(\frac{\partial}{\partial x^j}) \, , \tag{6}$$

where $(e^1, \ldots, e^n)_x$ is the dual frame to $u = (e_1, \ldots, e_n)_x$.

Secondly consider the *frame* or *n-symplectic velocity* coordinates (x^i, v^i_j) on $\lambda_M^{-1}(U)$ defined by

$$x^i(u) = x^i(x) \ , \qquad v^i_j(u) = e_j(x^i) \, , \tag{7}$$

The relationship between the two coordinates systems on LM is given by

$$v^i_j(u)\pi^j_k(u) = \delta^i_k \ \ , \qquad v^i_j(u)\pi^k_i(u) = \delta^k_j \, , \tag{8}$$

for all u in the domain of the π^i_j momentum coordinates.

Denoting the standard basis of $gl(n, \mathbb{R})$ by $\{E^i_j\}$, the corresponding fundamental vertical vector fields E^{*i}_j on LM are given in momentum coordinates by

$$E^{*i}_j = -\pi^i_k \frac{\partial}{\partial \pi^j_k} \, . \tag{9}$$

The bundle of linear frames LM is an open and dense submanifold of the n-tangent bundle $T^1_n M$, where $n = \dim M$. The general linear group $GL(n, \mathbb{R})$ acts naturally on both LM and $T^1_n M$. However, since each point in LM is a linear frame, the action of $Gl(n, \mathbb{R})$ is free on LM but not on $T^1_n M$. This reflects the fact that LM has more intrinsic structure than $T^1_n M$.

On LM we have an $\mathbb{R}^n$-valued one-form, the *soldering one-form* $\hat{\theta} = \theta^i \, \hat{r}_i$. Here $\hat{r}_i$ denotes the standard basis of $\mathbb{R}^n$. In momentum coordinates, θ^i has the local expression

$$\theta^i = \pi^i_j dx^j \, . \tag{10}$$

$\hat{\theta}$ is the n-symplectic potential on LM.

Now the restriction of the n-tangent structure on $T^1_n M$ to LM will yield an n-tangent structure on LM. It is not difficult to show that the restriction of (5) to LM has, in n-symplectic momentum coordinates, the form

$$J^i = -\pi^i_a \pi^j_b \frac{\partial}{\partial \pi^a_j} \otimes dx^b \, , \tag{11}$$

We will present now an alternative derivation of this n-tangent structure on LM that is reminiscent of the geometric origins of other tangent-like structures. We recall the formula

$$\xi^*(u) = \frac{d}{dt}(u \cdot \exp(t\xi))|_{t=0} \tag{12}$$

for the value of the associated fundamental vertical vector field ξ^* on LM defined at $u = (x, e_i)$ for each $\xi \in gl(n, \mathbb{R})$. These vector fields are smooth. We define the vector-valued 1-forms J^i by

$$(J^i)_u(X) = (E^i_j \theta^j_u(X))^*(u) \quad \forall \; X \in T_u(LM) \tag{13}$$

This definition uses the group action on LM in a manner that parallels the definition of the tangent structure on TM and mixes in the canonical soldering 1-forms in a fundamental way. The difference is that the action of $GL(n, \mathbb{R})$ on LM is global, while the definition of J on TM uses the fiberwise action of $T_n M$ on $T_n M$.

The mapping $\xi \to \xi^*$ is a linear mapping from the Lie algebra $gl(n, \mathbb{R})$ to the Lie algebra of fundamental vertical vector fields on LM. Hence

$$(J^i)_u(X) = \theta^j_u(X)(E^i_j)^*(u) \quad \forall \; X \in T_u(LM)$$

so that

$$J^i = E_j^{*i} \otimes \theta^j \tag{14}$$

Substituting (9) and (10) into this formula yields the local expression (11). This formula tell us that the canonical n-tangent structure on $T_n^1 M$ is in fact another representation of the soldering 1-form $\hat{\theta}$. To see this explicity we note that the mapping

$$\hat{r}_i \to E_i^j \otimes \hat{r}_j \to E_i^{*j} \otimes \hat{r}_i$$

is a linear representation of the basis vectors $\hat{r}_i$ of $\mathbb{R}^n$ in the space of $gl(n, \mathbb{R}) \otimes \mathbb{R}^n$. Extending this representation to $\hat{\theta} = \theta^i \otimes \hat{r}_i$ we obtain the n-tangent structure $\hat{J}$:

$$\hat{\theta} = \theta^i \otimes \hat{r}_i \to (E_j^{*i} \otimes \theta^j) \otimes \hat{r}_i = \hat{J}\,.$$

2.4 The vector-valued one-form S_α on $J^1\pi$

We now turn our attention to 1-jets and review the tangent-like structure present on $J^1\pi$ [17].

Let $\pi : E \to M$ be a fiber bundle where M is n-dimensional and E is $m = (n+k)$-dimensional. Let $\tau_E\rfloor_{V\pi} : V\pi \to E$ be the vertical tangent bundle to π. We shall denote by $\pi_{1,0} : J^1\pi \to E$ the canonical projection and by $V\pi_{1,0}$ the vertical distribution defined by $\pi_{1,0}$.

Throughout this paper if (x^i, y^A) are local fiber coordinates on E we take standard jet coordinates (x^i, y^A, y_i^A), $1 \leq i \leq n, 1 \leq A \leq k$, on the first jet bundle $J^1\pi$ the manifold of 1-jets of sections of π.

Definition 2.5 *Let $\phi : M \to E$ be a section of π, $x \in M$ and $y = \phi(x)$. The* vertical differential *of the section ϕ at the point $y \in E$ is the map*

$$\begin{array}{rcccc} d_y^V\phi & : & T_yE & \longrightarrow & V_y\pi \\ & & u & \mapsto & u - (\phi \circ \pi)_* u \end{array}$$

As $d_y^V\phi$ depends only on $j_x^1\phi$, the vertical differential can be lifted to $J^1\pi$ in the following way.

Definition 2.6 *The* canonical contact 1-*form* ω^1 *on* $J^1\pi$ *is the* $V\pi$*-valued* 1*-form defined by*

$$\begin{array}{rcccc} \omega^{\mathbf{1}}(j^1_x\phi) & : & T_{j^1_x\phi}(J^1\pi) & \longrightarrow & V_{\phi(x)}\pi \\ & & \tilde{X}_{j^1_x\phi} & \mapsto & (d^V_y\phi)\left((\pi_{1,0})_*(\tilde{X}_{j^1_x\phi})\right) \end{array}$$

In coordinates,

$$\omega^{\mathbf{1}} = (dy^B - y^B_j\,dx^j)\otimes\frac{\partial}{\partial y^B} \tag{15}$$

Next let us recall the definition of the vector-valued 1-form S_α on $J^1\pi$ where α is a 1-form on M. Given a point $j^1_x\phi\in J^1\pi$, a cotangent vector $\eta_x\in T^*_xM$ and a tangent vector $\xi\in V_{\phi(x)}\pi$, there exists a well defined vector $\eta_x\odot_{j^1_x\phi}\xi\in V_{j^1_x\phi}\pi_{1,0}$ called the *vertical lift* of ξ to $V\pi_{1,0}$ by η. This vector is locally given by

$$\eta_x\odot_{j^1_x\phi}\xi_{\phi(x)} = \eta_i\,\xi^A\frac{\partial}{\partial v^A_i}(j^1_x\phi)\,. \tag{16}$$

Definition 2.7 *Let* $\alpha\in\Lambda^1M$ *be any* 1*-form on* M*. The vector-valued* 1*-form* S_α *on* $J^1\pi$ *is defined by*

$$\begin{array}{rcl} S_\alpha(j^1_x\phi) : T_{j^1_x\phi}(J^1\pi) & \longrightarrow & (V\pi_{1,0})_{j^1_x\phi} \\ \tilde{X}_{j^1_x\phi} & \to & S_\alpha(j^1_x\phi)(\tilde{X}_{j^1_x\phi}) = \alpha_x\odot_{j^1_x\phi}\omega^{\mathbf{1}}(\tilde{X}_{j^1_x\phi})\,. \end{array}$$

From (16) and (15) we have that in coordinates

$$S_\alpha = \alpha_j\,(dy^A - y^A_i\,dx^i)\otimes\frac{\partial}{\partial v^A_j}\,. \tag{17}$$

S_α can be considered a more general version of the canonical tangent and k-tangent structures. This relationship is explored in section 3.2. Note also that S_α plays an important role in the construction of the Cartan-Hamilton-Poincaré n-form (see section 7.1).

2.5 The adapted frame bundle $L_\pi E$

An *adapted frame* at $e\in E$, $\pi : E\to M$, is a frame where the last k basis vectors are vertical with respect to π. The *adapted frame bundle* of π [28, 29], denoted by $L_\pi E$, consists of all adapted frames for E,

$$L_\pi E = \{(e_i, e_A)_e : e\in E, \{e_i, e_A\} \text{ is a basis for } T_eE, \text{ and } \pi_*(e)(e_A) = 0\}$$

The canonical projection, $\lambda : L_\pi E \to E$, is defined by $\lambda(e_i, e_A)_e = e$.

$L_\pi E$ is a reduced subbundle of LE, the frame bundle of E. As such it is a principal fiber bundle over E. Its structure group is $\mathrm{G_v}$ the nonsingular block lower triangular matrices

$$\mathrm{G_v} = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in Gl(n,\mathbb{R}), B \in Gl(k,\mathbb{R}), C \in \mathbb{R}^{kn} \right\} \tag{18}$$

$\mathrm{G_v}$ acts on $L_\pi E$ on the right by

$$(e_i, e_A)_e \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = (A^i_j e_i + C^A_j e_A, B^A_B e_A)_e. \tag{19}$$

If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $\lambda^{-1}(U)$. First consider the *coframe* or *m-symplectic momentum* coordinates $(x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)$ on $\lambda^{-1}(U)$ defined in (6). Let us observe that $\pi^i_A = 0$ on $L_\pi E$.

We have as is customary retained the same symbols for the induced horizontal coordinates.

Secondly consider the *frame* or *m-symplectic velocity* coordinates $(x^i, y^A, v^i_j, v^A_j, v^A_B)$ on $\lambda^{-1}(U)$ defined in (7). Let us observe that $v^i_A = 0$ on $L_\pi E$.

The v coordinates, viewed together as a block triangular matrix, form the inverse of the π coordinates defined above. The blocks have the following relations:

$$v^i_j \pi^j_s = \delta^i_s \qquad v^A_j \pi^j_s + v^A_B \pi^B_s = 0 \qquad v^A_B \pi^B_C = \delta^A_C$$

Lastly consider the following coordinates which are constructed from the previous two. Define $(x^i, y^A, u^i_j, u^A_j, u^A_B)$ on $\lambda^{-1}(U)$ by

$$x^i((e_i, e_A)_e) = x^i(e) \qquad u^i_j = \pi^i_j \qquad u^A_j = v^A_i \pi^i_j = -v^A_B \pi^B_j$$
$$y^A((e_i, e_A)_e) = y^A(e) \qquad u^A_B = \pi^A_B$$

In Section 3.3 we discuss the fact that $L_\pi E$ is an $H = Gl(n) \times Gl(k)$ principal bundle $\rho : L_\pi E \to J^1\pi$. It will turn out that the u^A_j coordinates are pull-ups under ρ of the standard jet coordinates on $J^1\pi$. As such, we refer to these coordinates as *Lagrangian* coordinates.

The fundamental vertical vector fields E^{*i}_j, E^{*A}_B and E^{*i}_A, on $L_\pi E$ are given, in Lagrangian coordinates, by

$$E^{*i}_j = -u^i_s \frac{\partial}{\partial u^j_s} \qquad E^{*A}_B = -u^A_C \frac{\partial}{\partial u^B_C} \qquad E^{*i}_A = u^i_s v^B_A \frac{\partial}{\partial u^B_s} \tag{20}$$

On $L_\pi E$ we have also a $\mathbb{R}^{m+k}$-valued 1-form, the *soldering one-form* $\hat\theta = \theta^i\,\hat r_i + \theta^A\,\hat r_A$, which is the restriction of the canonical soldering 1-form on LE to $L_\pi E$. Here $\hat r_i, \hat r_A$ denotes the standard basis of $\mathbb{R}^{n+k}$. In Lagrangian coordinates, θ^i, θ^A have the local expression

$$\theta^i = u^i_j dx^j\,, \quad \theta^A = u^A_B(dy^B - u^B_j dx^j)\,. \tag{21}$$

From (14) we have that the $(n+k)$-tangent structure on LE is given by

$$J^i = E^{*i}_j \otimes \theta^j + E^{*i}_B \otimes \theta^B, \quad J^A = E^{*A}_j \otimes \theta^j + E^{*A}_B \otimes \theta^B$$

Now considering its restriction to the principal fiber bundle $L_\pi E$ we have

$$(J^i)|_{L_\pi E} \equiv J^i, \qquad 1 \leq i \leq n,$$

$$J^A|_{L_\pi E} \equiv E^{*A}_j|_{L_\pi E} \otimes \theta^j + E^{*A}_B \otimes \theta^B \qquad 1 \leq A \leq k\,.$$

3 Relationships among the tangent-like structures

In this section we show how the tangent, k-tangent, and similar structures on various spaces are related. We have already remarked in Section 2.3 that the n-tangent structure on LM and the one on $T^1_n M$ ($n = \dim M$) induce each other. Now we complete the circle by showing that the tangent structure on TM induces the k-tangent structure on $T^1_k M$ and that the n-tangent structure on LM induces the tangent structure on TM.

Secondly, we show that in the special cases where comparison makes sense the vector valued one-form on $J^1\pi$ is directly related to the k-tangent structure on $T^1_k M$. Furthermore, using recent results relating the jet bundle and adapted frame bundle, we show a similar relationship with the $(n+k)$-tangent structure on $L_\pi E$.

3.1 Relationships among TM, $T^1_k M$, and LM

<u>The k-tangent structure on $T^1_k M$ in terms of the tangent structure on TM</u>

One can induce J^A on $T^1_k M$ from J on TM. We make use of the *inclusion maps*

$$\begin{array}{rccc} i_A: & TM & \to & T^1_k M \qquad\qquad 1 \leq A \leq k \\ & v_x & \to & (0,\dots,0,v_x,0,\dots,0) \end{array}$$

From (1), (5) we obtain

Proposition 3.1

$$J^A(u) = i_{A*}(\phi(u)) \circ J_{\phi(u)} \circ \phi_*(u)$$

for all $u \in T^1_k M$, *where* $\phi : T^1_k M \to TM$ *is any* C^1 *bundle morphism over the identity on* M*(*one of the k projections for example*).*

<u>The tangent structure on TM viewed from LM</u>

Lemma 3.2 *Let* $(J^1, \ldots, J^n)$ *be the canonical* n*-tangent structure of* LM*. For all vector fields* X *on* LM *we have*

$$J^i \circ R_{g*}(X) = (g^{-1})^i_a R_{g*} \circ J^a(X) \tag{22}$$

where R_g denotes the right translation with respect to $g \in GL(n, \mathbb{R})$.

Proof It follows from (11) and the identities

$$R^*_g(\pi^j_k) = (g^{-1})^j_a \pi^a_k, \quad R^*_g(d\pi^j_k) = (g^{-1})^j_a d\pi^a_k, \quad R^*_g(dx^i) = dx^i\,,$$

$$R_{g*}(\frac{\partial}{\partial \pi^i_j}) = (g^{-1})^a_i \frac{\partial}{\partial \pi^a_j}, \quad R_{g*}(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}\,.$$

where g is any element of $GL(n, \mathbb{R})$. ■

Let $\tilde{T}M$ denotes the manifold obtained from the tangent bundle TM by deleting the zero section. For a fixed, non-zero element $\xi \in \mathbb{R}^n$ let ψ_ξ denote the mapping from LM to $\tilde{T}M$ defined as follows. For each $u \in LM$ let

$$\psi_\xi(u) = [u, \xi] \tag{23}$$

where we are identifying the tangent bundle TM with the bundle associated to LM and the standard action of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$. The following lemma is easily verified for this map.

Lemma 3.3

$$(\psi_\xi)_*(E^{*b}_c(u)) = \xi^b v^i_c(u) \left.\frac{\partial}{\partial y^i}\right|_{[u,\xi]} \tag{24}$$

Remark In this case Let $h = h_{ij} dx^i \otimes dx^j$ be any metric tensor field on the manifold M. Then its associated covariant tensorial function on LM is the $\mathbb{R}^{n*} \otimes_s \mathbb{R}^{n*}$ valued function with components

$$\hat{h}_{ij} = (h_{ab} \circ \lambda) v_i^a v_j^b \tag{25}$$

(see [30]). For simplicity we will drop the $\circ\lambda$ notation and write simply $\hat{h}_{ij} = h_{ab} v_i^a v_j^b$. Moreover, we know that $(\hat{h}_{ab})$ obeys the transformation law

$$\hat{h}_{ab}(u \cdot g) = g_a^m g_b^n \hat{h}_{mn}(u) \tag{26}$$

for all $g \in GL(n)$.

Definition 3.4 *Let h be a fixed positive definite metric tensor field on M. The associated* covariant m-tangent structure $(J_i^{(h)})$ based on h *is*

$$J_i^{(h)} = \sum_j^m \hat{h}_{ij} J^j \tag{27}$$

Lemma 3.5

$$J_a^{(h)}(u \cdot g)(R_{g*}(X)) = g_a^b R_{g*}\left(J_b^{(h)}(u)(X)\right) \tag{28}$$

Proof The proof follows easily from (22) and (26). ■

Theorem 3.6 *Let h be an arbitrary positive definite metric tensor field on the manifold M, and let $(J_i^{(h)})$ denote the covariant m-tangent structure on LM defined by h. For each point $[u, \xi] \in \tilde{T}M$ (note: $\xi = (\xi^i)$ is by assumption non-zero) let $\psi_\xi : LM \to \tilde{T}M$ be the map defined in (23) above. Then the vector-valued 1-form $\mathcal{J}$ on $\tilde{T}M$ defined pointwise by*

$$X \longrightarrow \mathcal{J}([u, \xi])(X) = \frac{\psi_{\xi*}\left(\xi^i J_i^{(h)}(u)(\tilde{X})\right)}{\hat{h}_{ab}(u)\xi^a \xi^b}, \qquad \forall\, X \in T_{[u,\xi]}(TM) \tag{29}$$

is the canonical tangent structure on $\tilde{T}M$ given in local coordinates by

$$\mathcal{J} = \frac{\partial}{\partial y^i} \otimes dx^i \tag{30}$$

In equation (29) $\tilde{X}$ is any tangent vector at $u \in LM$ that projects to the same vector at $\lambda_M(u)$ as does the vector $X \in T_{[u,\xi]}(TM)$; i.e. $d\lambda_M(\tilde{X}) = d\tau(X)$.

Proof We first show that the tangent vector $\mathcal{J}([u,\xi])$ is well-defined. Since $[u,\xi] = [u\cdot g, g\cdot\xi]$ we need to show that the right-hand side of formula (29) remains unchanged if we make the substitutions $u \to u\cdot g$ and $\xi \to g\cdot\xi = ((g^{-1})^i_j\xi^j)$. Making the substitutions we have

$$\mathcal{J}([u\cdot g, g\,\xi])(X) = \frac{\psi_{(g\,\xi)*}\left((g\,\xi)^i J_i^{(h)}(u\cdot g)(R_{g*}\tilde{X})\right)}{\hat{h}_{ab}(u\cdot g)(g\cdot\xi)^a(g\cdot\xi)^b} \tag{31}$$

Using $(g\,\xi^i) = (g^{-1})^i_m\xi^m$ and (28) the numerator in this equation can be reduced as follows:

$$\begin{aligned}
\psi_{(g\,\xi)*}\left((g\,\xi)^i J_i^{(h)}(u\cdot g)(R_{g*}\tilde{X})\right) &= \psi_{(g\,\xi)*}\left((g^{-1})^i_m\xi^m g^b_i R_{g*}\left(J_b^{(h)}(u)(X)\right)\right)\\
&= \psi_{(g\,\xi)*}\left(R_{g*}(\xi^i J_i^{(h)}(u)(\tilde{X})\right)\\
&= (\psi_{(g\,\xi)}\circ R_g)_*\left(\xi^i J_i^{(h)}(u)(\tilde{X})\right)\\
&= \psi_{\xi*}\left(\xi^i J_i^{(h)}(u)(\tilde{X})\right)
\end{aligned}$$

where the last equality follows from the fact that $\psi_{g\,\xi}\circ R_g = \psi_\xi$.

Similarly, using (26) the denominator in equation (31) can be reduced as follows:

$$\hat{h}_{ab}(u\cdot g)(g\cdot\xi)^a(g\cdot\xi)^b = \hat{h}_{ab}(u)\xi^a\xi^b$$

Substituting the last two results into (31) we obtain

$$\frac{\psi_{(g\cdot\xi)*}\left((g\,\xi)^i J_i^{(h)}(u\cdot g)(R_{g*}\tilde{X})\right)}{\hat{h}_{ab}(u\cdot g)(g\cdot\xi)^a(g\cdot\xi)^b} = \frac{\psi_{\xi*}\left(\xi^i J_i^{(h)}(u)(\tilde{X})\right)}{\hat{h}_{ab}(u)\xi^a\xi^b}$$

which proves that the mapping given in (29) above is well-defined.

We now calculate the numerator on the right-hand-side of the above identity. From (24), (27), we obtain

$$\begin{aligned}
\psi_{\xi*}\left(\xi^i J_i^{(h)}(u)(\tilde{X})\right) &= \left(\xi^i\hat{h}_{ij}(u)\theta^k(u)(\tilde{X})\right)\psi_{\xi*}\left(E_k^{*j}(u)\right)\\
&= \left(\xi^i\hat{h}_{ij}(u)\pi^k_l(u)dx^l(\tilde{X})\right)\left(\xi^j v^a_k(u)\frac{\partial}{\partial y^A}([u,\xi])\right)\\
&= \left(\hat{h}_{ij}(u)\xi^i\xi^j\right)\left(\frac{\partial}{\partial y^A}([u,\xi])dx^a(X)\right)\\
&= \left(\hat{h}_{ij}(u)\xi^i\xi^j\right)\left(\frac{\partial}{\partial y^A}\otimes dx^a\right)([u,\xi])(X)
\end{aligned}$$

Since the metric h is definite, the coefficient $\hat{h}_{ij}(u)\xi^i\xi^j$ is non-zero for all $u\in LM$. Hence we may divide both sides of the last equation by this term and use linearity of the mapping ψ_ξ to obtain the desired result. ■

3.2 The relationship between the vertical endomorphism on $J^1\pi$ and the canonical k-tangent structures

Now we shall describe the relationship between the vertical endomorphism on $J^1\pi$ and the canonical k-tangent structure on T^1_kM when E is the trivial bundle $E = \mathbb{R}^k \times M \to \mathbb{R}^k$. In this case $J^1\pi$ is diffeomorphic to $\mathbb{R}^k \times T^1_kM$ via the diffeomorphism given by $j^1_t\phi \equiv (t, j^1_0\phi_t)$ where $\phi_t(s) = \phi(t+s)$. In this case, (see (17)), the vector valued 1-form S_α is locally given by

$$S_\alpha = \frac{\partial}{\partial v^i_B} \otimes \left(\alpha_B \left(dx^i - v^i_A\, dt^A\right)\right)$$

with respect the coordinates (t^A, x^i, v^i_A) on $\mathbb{R}^k \times T^1_kM$.

In the case $k = 1$, we consider $\omega = dt$ and thus

$$S_{dt} = \frac{\partial}{\partial v^i} \otimes (dx^i - v^i\, dt) = \frac{\partial}{\partial v^i} \otimes dx^i - v^i \frac{\partial}{\partial v^i} \otimes dt$$

where (t, x^i, v^i) are the coordinates in $\mathbb{R}^n \times TM$. Then we have

$$S_{dt} = J - C \otimes dt$$

where C denotes the canonical or Liouville vector field on TM and J is the canonical tangent structure J on TM.

In the general case, with k arbitrary, if we fix B, $1 \leq B \leq k$, we have

$$S_{dt^B} = \frac{\partial}{\partial v^i_B} \otimes (dx^i - v^i_A\, dt^A) = \frac{\partial}{\partial v^i_B} \otimes dx^i - v^i_A \frac{\partial}{\partial v^i_B} \otimes dt^A = J^B - C^B_A \otimes dt^B$$

where J^B is the canonical k-tangent structure on T^1_kM, and the C^B_A are the *canonical vertical vector fields* defined in equation (4).

Proposition 3.7 *The relationship between the canonical k-tangent structure on T^1_kM and the vertical endomorphism S_{dt^B}, up to some obvious identifications, is given by*

$$J^B = S_{dt^B} + C^B_A \otimes dt^A$$

3.3 Strong relationships between $J^1\pi$ and $L_\pi E$

We shall consider two ways of describing 1-jets, each with its own charm:

1. Equivalence classes of local sections of π.

$$J^1\pi = \{j_x^1\phi : x \in M, \phi \in \Gamma_x(\pi)\}$$

where $\Gamma_x(\pi)$ denotes the set of sections of π defined in a neigboorhood of x.

2. Linear right-inverses to $\pi_*(e)$.

$$J^1\pi = \{\tau_e : T_{\pi(e)}M \to T_eE : \pi_*(e) \circ \tau_e = id_{T_{\pi(e)}M}\}$$

We will use either description of $J^1\pi$ when it is convenient.

Let H be the subgroup of $\mathrm{G_v}$ isomorphic to $Gl(n) \times Gl(k)$ (defined in (18)) given by

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in Gl(n, \mathbb{R}), B \in Gl(k, \mathbb{R}) \right\} \quad ,$$

and let $\mathcal{J}$ be the following subgroup of $\mathrm{G_v}$

$$\mathcal{J} = \left\{ \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} : \xi \in \mathbb{R}^{kn} \right\} \quad .$$

Although H is a closed Lie subgroup of G_V, it is not normal. As such $\mathrm{G_v}/H$ does not have a natural group structure; it is a manifold with a left $\mathrm{G_v}$-action. For each coset $gH \in \mathrm{G_v}/H$, we select the unique representative in $\mathcal{J}$.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$$

By choosing these representatives, we identify $\mathrm{G_v}/H$ with $\mathcal{J}$ and hence $\mathbb{R}^{kn}$. These identifications are diffeomorphisms.

Consider how the left $\mathrm{G_v}$-action looks for our selected representatives.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C + B\xi & B \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ CA^{-1} + B\xi A^{-1} & I \end{pmatrix} \tag{32}$$

So the $\mathrm{G_v}$-action appears *affine* when $\mathrm{G_v}/H$ is identified with $\mathbb{R}^{kn}$. Therefore it is prudent to use this identification to define an affine structure on $\mathrm{G_v}/H$ modeled on $\mathbb{R}^{kn}$. This $\mathrm{G_v}$-invariant structure will pass to the fibers of the associated bundle discussed below, making it an affine bundle.

Theorem 3.8 $L_\pi E \times_{\mathrm{G_v}} (\mathrm{G_v}/H) \cong J^1\pi$

Proof: The affine bundle isomorphism maps each equivalence class $[(e_i, e_A)_e, (\xi_i^A)]$ to the linear map $\phi : T_{\pi(e)}M \to T_eE$ defined by $\phi(\hat{e}_i) = e_i + \xi_i^A e_A$, where we use the basis $\{\hat{e}_i\}$ where $\hat{e}_i = \pi_*(e)(e_i)$. The inverse isomorphism is given by

$$j_x^1\phi \longmapsto [\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)_{\phi(x)}, \left(\frac{\partial \phi^a}{\partial x^i}(x)\right)]$$

The following corollary, whose simple proof is made possible by the preceding development, is a fundamental tool in lifting Lagrangian field theory to the adapted frame bundle.

Corollary 3.9 *$L_\pi E$ is a principal fiber bundle over $J^1\pi$ with structure group H.*

Proof: This fact follows directly from proposition 5.5 in reference [30]. ■

We will denote the projection from $L_\pi E$ to $J^1\pi$ by ρ. It is given by

$$\begin{array}{rllllll} \rho : & L_\pi E & \longrightarrow & J^1\pi & & & \\ & (e_i, e_A)_e & \longmapsto & \tau_e : & T_{\pi(e)}M & \to & T_eE \\ & & & & \pi_*(e)(e_i) & \mapsto & e_i \end{array}$$

We now show that the u_j^A-coordinates defined in Section 2.5 are the pull-ups of the jet coordinates. If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$ and $u = (e_i, e_A)_e \in \lambda^{-1}(U)$ then

$$\begin{aligned} y_i^A \circ \rho(u) = y_i^A(\tau_e) = (dy^A)_e \left(\tau_e(\frac{\partial}{\partial x^i}\Big|_{\pi(e)})\right) = (dy^A)_e(\hat{e}^j(\frac{\partial}{\partial x^i}\Big|_{\pi(e)})e_j) \\ = e^j(\frac{\partial}{\partial x^i}\Big|_e)(dy^A)_e(e_j) = \pi_i^j(u)v_j^A(u) = u_j^A(u) \end{aligned}$$

3.4 The vector-valued 1-form S_α on $J^1\pi$ viewed from $L_\pi E$.

$L_\pi E$ is a principal bundle over $J^1\pi$, we shall establish in this subsection the relationship between the vertical endomorphism S_α on $J^1\pi$ and the restriction of the $(n+k)$-tangent structure of LE to the vertical adapted bundle $L_\pi E$. To be more precise, we show that S_α corresponds to the tensors on $L_\pi E$:

$$E_B^{*i} \otimes \theta^B = J^i - E_j^{*i} \otimes \theta^j, \quad 1 \le i \le n\,.$$

Note the similarity to proposition 3.7.

Proposition 3.10 *Let $u = (e_i, e_A)_e$ be a frame on $L_\pi E$ and let us denote by $u \cdot \xi$ the frame*

$$u \cdot \xi \quad = \quad (e_i, e_A)_e \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} \quad = \quad (e_i + \xi_i^B\, e_B, e_A)_e$$

Let α be any 1-form on M and $[u, \xi] = [(e_i, e_A)_e, (\xi_i^A)]$ an element of $J^1\pi$. Then the relationship between S_α and the tensor fields $E_B^{*i} \otimes \theta^B$ is given by

$$S_\alpha([u, \xi])(X_{[u,\xi]}) \;=\; \rho_*(u \cdot \xi)\left((\pi^*\alpha)_e(e_i)\,(E_B^{*i} \otimes \theta^B)(u \cdot \xi)(\tilde{X}_{u \cdot \xi})\right) \tag{33}$$

where

$$X_{[u,\xi]} \in T_{[u,\xi]}(J^1\pi)\,, \quad \tilde{X}_{u \cdot \xi} \in T_{u\,\xi}(L_\pi E)$$

are vectors that project onto the same vector on E.

Proof : First let us observe that, from the definition of ρ, we have

$$\rho(u \cdot \xi) = \rho((e_i + \xi_i^B\, e_B, e_A)_e) = [(e_i, e_A)_e, (\xi_i^A)] = [u, \xi] \quad .$$

Now we shall prove that the right side of (33) does not depend on the choice of the representative of the equivalence class $[(e_i, e_A)_e, (\xi_i^A)]$. If

$$[u, \xi] = [(e_i, e_A)_e, (\xi_i^A)] \;=\; [(\bar{e}_i, \bar{e}_A)_e, (\bar{\xi}_i^A)] \;=\; [\bar{u}, \bar{\xi}]$$

we must prove that

$$\rho_*(u \cdot \xi)\left((\pi^*\alpha)_e(e_i)\,(E_B^{*i} \otimes \theta^B)(u\,\xi)(X_{u \cdot \xi})\right) \;=\; \rho_*((\bar{u} \cdot \bar{\xi})\left((\pi^*\alpha)_e(\bar{e}_j)\,(E_C^{*j} \otimes \theta^C)((\bar{u} \cdot \bar{\xi})(\bar{X}_{\bar{u} \cdot \bar{\xi}})\right)$$

for any vectors $X_{u \cdot \xi} \in T_{u \cdot \xi}(L_\pi E)$, $\bar{X}_{\bar{u} \cdot \bar{\xi}} \in T_{\bar{u} \cdot \bar{\xi}}(L_\pi E)$ that project at the same vector on E.

But, in this case, we have from (19) and (32)

$$\bar{u} = (\bar{e}_j, \bar{e}_B)_e \;= (A_j^i\, e_i + C_j^A\, e_A\,, B_B^A\, e_A), \quad \bar{\xi}_j^B = -(B^{-1})_C^B\, C_j^C + (B^{-1})_C^B\, \xi_i^C\, A_j^i \quad . \tag{34}$$

Let us consider the frames

$$\tilde{u} = (\tilde{e}_i, \tilde{e}_A)_e = u \cdot \xi = (e_i + \xi_i^B\, e_B, e_A)_e$$

$$\hat{u} = (\hat{e}_j, \hat{e}_B)_e = \bar{u} \cdot \bar{\xi} = (\bar{e}_j + \bar{\xi}_j^C\, \bar{e}_C, \bar{e}_B)_e = (A_j^i\, \tilde{e}_i, B_B^A\, \tilde{e}_A)_e$$

where the last identity comes from (34). Then we deduce that the relationship between the coordinates of $\tilde{u}$ and $\hat{u}$ are

$$\hat{v}_j^l = A_j^i\,\tilde{v}_i^l\,,\ \hat{v}_j^C = A_j^i\,\tilde{v}_i^C\,,\ \hat{v}_B^C = B_B^A\,\tilde{v}_A^C\,,\ \hat{u}_j^i = (A^{-1})_l^i\,\tilde{u}_j^l\,,\ \hat{u}_l^A = \tilde{u}_l^A\ . \tag{35}$$

On the other hand, the tensor fields $E_B^{*i}\otimes\theta^B$ are locally given by

$$E_B^{*i}\otimes\theta^B = u_j^i\,(dy^B - u_t^B\,dx^t)\otimes\frac{\partial}{\partial u_j^B}\quad . \tag{36}$$

From (36), and (35) we obtain

$$(E_C^{*j}\otimes\theta^C)(\hat{u}) = (A^{-1})_r^j\,\tilde{u}_l^r\,((dy^A)_{\hat{u}} - \tilde{u}_t^A\,(dx^t)_{\hat{u}})\otimes\frac{\partial}{\partial u_l^A}(\hat{u}) \tag{37}$$

Since $(\pi^*\alpha)_e(\bar{e}_j) = A_j^i\,(\pi^*\alpha)_e(e_i)$ we deduce that

$$(\pi^*\alpha)_e(\bar{e}_j)\,(E_C^{*j}\otimes\theta^C)(\hat{u}) \;=\; (\pi^*\alpha)_e(e_i)\,\tilde{u}_l^i\,((dy^A)_{\hat{u}} - \tilde{u}_t^A\,(dx^t)_{\hat{u}})\otimes\tfrac{\partial}{\partial u_l^a}(\hat{u}) \tag{38}$$

and

$$(\pi^*\alpha)_e(e_i)(E_B^{*i}\otimes\theta^B)(\tilde{u}) = (\pi^*\alpha)_e(e_i)\,\tilde{u}_j^i\,((dy^A)_{\tilde{u}} - \tilde{u}_l^A\,(dx^l)_{\tilde{u}})\otimes\frac{\partial}{\partial u_j^A}(\tilde{u}) \tag{39}$$

If the vectors $X_{u\cdot\xi}\in T_{u\cdot\xi}(L_\pi E)$, $\bar{X}_{\bar{u}\cdot\bar{\xi}}\in T_{\bar{u}\cdot\bar{\xi}}(L_\pi E)$ project onto the same vector on E then its components with respect the coordinates x^i and y^A are equal and from (38) and (39) we obtain that

$$\rho_*(\tilde{u})\left((\pi^*\alpha)_e(e_i)(E_B^{*i}\otimes\theta^B)(\tilde{u})(X_{\tilde{u}})\right) = \rho_*(\hat{u})\left((\pi^*\alpha)_e(\bar{e}_j)(E_C^{*j}\otimes\theta^C)(\hat{u})(\bar{X}_{\hat{u}})\right)$$

because $\rho(\tilde{u}) = [u,\xi] = [\bar{u},\bar{\xi}] = \rho(\hat{u})$.

Now we shall prove the identity (33) using the Theorem 3.8 . If $j_x^1\phi\equiv[(e_i,e_A)_e,(\xi_i^A)]$ and

$$e_i = v_A^j\,\frac{\partial}{\partial x^j}(e) + v_i^B\,\frac{\partial}{\partial y^B}(e)\quad ,\quad e_A = v_A^B\,\frac{\partial}{\partial y^B}(e) \tag{40}$$

then

$$[(e_i,e_A)_e,(\xi_i^A)] = [(\frac{\partial}{\partial x^j},\frac{\partial}{\partial y^B})_e,\begin{pmatrix} v_A^j & 0\\ v_i^B & v_A^B\end{pmatrix}|_e(\xi_i^A)]$$

From this identity and (32) we deduce that the coordinates of the 1-jet $j^1_x\phi$, defined by the class $[u,\xi] = [(e_i, e_A)_e, (\xi^A_i)]$, are

$$\frac{\partial \phi^B}{\partial x^s}(\pi(e)) = u^r_s\,(v^B_r + v^B_A\,\xi^A_r)|_e \tag{41}$$

and therefore from (17) we have

$$S_\alpha([u,\xi]) = \alpha_j(\pi(e))\,((dy^A)_{[u,\xi]} - u^r_i\,(v^A_r + v^A_B\,\xi^B_r)|_e\,(dx^i)_{[u,\xi]}) \otimes \frac{\partial}{\partial v^A_j}([u,\xi]) \tag{42}$$

The coodinates of the frame $\tilde{u}$ satisfy the identities

$$\tilde{u}^i_j = u^i_j \quad , \quad \tilde{u}^A_t = \tilde{u}^l_t\,\tilde{v}^A_l = u^l_t\,(v^A_l + v^A_B\,\xi^B_l) \quad .$$

Therefore, from (36) we have

$$(E^{*i}_B \otimes \theta^B)(\tilde{u}) = u^i_j|_e\,((dy^A)_{\tilde{u}} - u^l_t|_e\,(v^a_l + v^A_B\,\xi^B_l)|_e\,(dx^t)_{\tilde{u}}) \otimes \frac{\partial}{\partial u^a_j}(\tilde{u}) \tag{43}$$

If $\alpha = \alpha_r\,dx^r$, then $(\pi^*\alpha)_e(e_i) = \alpha_r(\pi(e))\,v^r_i$ and from (43) we obtain that

$$(\pi^*\alpha)_e(e_i)\,(E^{*i}_B \otimes \theta^B)(\tilde{u}) = \alpha_j\,((dy^A)_{\tilde{u}} - u^l_t|_e\,(v^a_l + v^A_B\,\xi^B_l)|_e\,(dx^t)_{\tilde{u}}) \otimes \frac{\partial}{\partial u^A_j}(\tilde{u}))$$

Now, since $\rho(\tilde{u}) = [u,\xi]$, from this last identity and (42) we get the identity (33) taking into account that $\rho^* y^A_j = u^A_j$. ∎

4 Spaces with cotangent-like structures

In this section we shall define and give the main properties of the almost cotangent structure and its generalizations.

4.1 Almost cotangent structures and T^*M

Almost cotangent structures were introduced by Bruckheimer [2]. An almost cotangent structure on a $2m$-dimensional manifold M consists of a pair (ω, V) where ω is a symplectic form and V is a distribution such that

$$(i) \quad \omega\rfloor_{V\times V} = 0, \qquad (ii) \quad \ker\omega = \{0\}$$

The canonical model of this structure is the cotangent bundle $\tau^*_M : T^*M \to M$ of an arbitrary manifold M, where ω is the canonical symplectic form $\omega_0 = -d\theta_0$ on T^*M and V is the vertical distribution. Let us recall the definition of the Liouville form θ_0 in T^*M:

$$\theta_0(\alpha)(\tilde{X}_\alpha) = \alpha((\tau^*_M)_*(\alpha)(\tilde{X}_\alpha)),$$

for all vectors $\tilde{X}_\alpha \in T_\alpha(T^*M)$. In local coordinates (x^i, p_i) on T^*M

$$\omega_0 = dx^i \wedge dp_i, \qquad V = \langle \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \rangle. \tag{44}$$

Clark and Goel [3] also investigated these structures, defining them as a certain type of G-structure. They proved that the integrability of these structures, that is the existence of coordinates on the manifold such that ω_0 and V have the form of (44), is characterized by

Proposition 4.1 *An almost cotangent structure (ω, V) on M is integrable if and only if ω is closed and the distribution V is involutive.* ■

Thompson [26, 31] proved that an integrable almost cotangent manifold M satisfiying some natural global hypotheses is essentially the cotangent bundle of some differentiable manifold.

4.2 k-symplectic structures and $(T^1_k)^*M$

Definition 4.2 *[7, 8] A k-*symplectic structure *on a manifold M of dimension $N = n + kn$ is a family $(\omega_A, V; 1 \leq A \leq k)$, where each ω_A is a closed 2-form and V is an nk-dimensional distribution on M such that*

$$(i) \quad \omega_{A_{\rfloor V\times V}} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker \omega_A = \{0\}.$$

*In this case (M, ω_A, V) is called a k-*symplectic manifold*.*

The canonical model of this structure is the k-cotangent bundle $(T^1_k)^*M = J^1(M, \mathbb{R}^k)_0$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with target at $0 \in \mathbf{R}^k$, and projection $\tau^*(j^1_{x,o}\sigma) = x$.

The manifold $(T^1_k)^*M$ can be canonically identified with the Whitney sum of k copies of T^*M, say

$$\begin{array}{rcl}(T^1_k)^*M & \equiv & T^*M \oplus \cdots \oplus T^*M, \\ j_{x,0}\sigma & \equiv & (j^1_{x,0}\sigma^1, \ldots, j^k_{x,0}\sigma^k)\end{array}$$

where $\sigma^A = \pi_A \circ \sigma : M \longrightarrow \mathbb{R}$ is the A-th component of σ.

The canonical k-symplectic structure $(\omega_A, V; 1 \leq A \leq k)$, on $(T^1_k)^*M$ is defined by

$$\omega_A = (\tau^*_A)^*(\omega_0)$$
$$V(j^1_{x,0}\sigma) = \ker(\tau^*)_*(j^1_{x,0}\sigma)$$

where $\tau^*_A = (T^1_k)^*M \to T^*M$ is the projection on the A^{th}-copy T^*M of $(T^1_k)^*M$, and ω_0 is the canonical symplectic structure of T^*M.

One can also define the 2-forms ω_A by $\omega_A = -d\theta_A$ where θ_A is the 1-form defined as follows

$$\theta_A(j^1_{x,0}\sigma)(\tilde{X}_{j^1_{x,0}\sigma}) = \sigma_*(x)((\tau^*_A)_*(j^1_{x,0}\sigma)\tilde{X}_{j^1_{x,0}\sigma})$$

for all vectors $\tilde{X}_{j^1_{x,0}\sigma} \in T_{j^1_{x,0}\sigma}(T^1_k)^*M$.

If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, p^A_i), $1 \leq i \leq n$, $1 \leq A \leq k$ on $(T^1_k)^*U = (\tau^*)^{-1}(U)$ are given by

$$x^i(j^1_{x,0}\sigma) = x^i(x), \qquad p^A_i(j^1_{x,0}\sigma) = d_x\sigma^A(\frac{\partial}{\partial x^i}\Big|_x) \, .$$

Then the canonical k-symplectic structure is locally given by

$$\omega_A = \sum_{i=1}^n dx^i \wedge dp^A_i, \qquad V = \langle \frac{\partial}{\partial p^1_i}, \ldots, \frac{\partial}{\partial p^k_i} \rangle \quad 1 \leq A \leq k \, .$$

Theorem 4.3 *[7] Let $(\omega_A, V; 1 \leq A \leq k)$ be a k-symplectic structure on M. About every point of M we can find a local coordinate system (x^i, p^A_i), $1 \leq i \leq n$, $1 \leq A \leq k$ such that*

$$\omega_A = \sum_{i=1}^n dx^i \wedge dp^A_i, \quad 1 \leq A \leq k \tag{45}$$

In [4] Günther introduces the following definitions.

Definition 4.4 *A closed non-degenerate* $\mathbb{R}^n$*-valued* 2*-form*

$$\bar{\omega} = \sum_{A=1}^{n} \omega_A \, \hat{r}_A$$

on a manifold M *of dimension* N *is called a* polysymplectic form. *The pair* $(M, \bar{\omega})$ *is a* polysymplectic manifold.

A polysymplectic form $\bar{\omega}$ *on a manifold* M *is called* standard *iff for every point of* M *there exists a local coordinate system such that* ω_A *is written locally as in (45).*

¿From Theorem 4.3 it now follows that the k-symplectic manifold structures coincide with the *standard polysymplectic* structures.

$\bar{\omega}$ is called by Norris [32] a general n-symplectic structure. The difference in the formalism is that there exist natural definitions of Poisson brackets in the n-symplectic theory of Norris. See Section 9 for a discussion of n-symplectic Poisson brackets in the general case.

4.3 Almost k-cotangent structures and $(T^1_k)^*M$

In [5] the almost k-cotangent structures were defined and described as G-structures.

Definition 4.5 *An* almost k-cotangent structure *is a family* $(\omega_A, V_A; 1 \leq A \leq k)$*, where each* ω_A *is a* 2*-form of constant rank* $2n$ *and* V_A *is a* n*-dimensional distribution on* M*, such that*

$$(i) \quad V_A \cap (\oplus_{B \neq A} V_B) = 0, \quad (ii) \quad \ker \omega_A = \oplus_{B \neq A} V_B, \quad (iii) \quad \omega_A \rfloor_{V_A \times V_A} = 0$$

for all $1 \leq A \leq k$*.*

The canonical model of this structure is $(T^1_k)^*M$ with the 2-forms ω_A, and $V_A = \ker T\rho_A$ where $\rho_A : (T^1_k)^*M \to (T^1_{k-1})^*M$ is the projection given by

$$\rho_A(\alpha_1, \dots, \alpha_k) = (\alpha_1, \dots, \alpha_{A-1}, \alpha_{A+1}, \dots, \alpha_k).$$

The integrability of these structures is characterized by

Proposition 4.6 *An almost* k*-cotangent structure* $(\omega_A, V_A; 1 \leq A \leq k)$ *on* M *is integrable if and only if the* 2*-forms* ω_A *are closed and all distributions* $V_{A_1} \oplus \cdots \oplus V_{A_k}$ *are involutive.*

Remark It can be proved that an integrable almost k-cotangent structure on a manifold M is a k-symplectic structure on M setting $V = \oplus_{A=1}^{k} V_A$.

4.4 The n-symplectic structure of LM

The frame bundle LM has a canonical n-symplectic structure given by $\omega_i = -d\theta^i$, $V = \ker \lambda_M$ where θ^i are the components of the *soldering one-form* and V is the vertical distribution. This structure was first introduced in [9, 10] under the name *generalized symplectic geometry on* LM, and later referred to as n-symplectic geometry in [11]. n-symplectic geometry is the generalized geometry that one obtains on LM when $d\hat{\theta} = d\theta^i \hat{r}_i$ is taken as a generalized symplectic 2-form. The structure is rich enough to allow the definition of generalized Poisson brackets and generalized Hamiltonian vector fields. The ideas are "generalized" in the sense that the observables of the theory are vector-valued on LM rather than $\mathbb{R}$-valued. Moreover the generalized Hamiltonian vector fields are equivalence classes of vector-valued vector fields. The details of this geometry in the more general case of a *general n-symplectic manifold* are given in Section 9 of this paper.

The relationship between n-symplectic geometry on the bundle of linear frames LM and canonical symplectic geometry on the cotangent bundle T^*M has been developed in [11], showing that the ordinary symplectic geometry of T^*M can be induced from the n-symplectic geometry of LM using the associated bundle construction. This relationship will be discussed further in Section 5.1.

In [28] it is shown that m-symplectic geometry on frame bundles can be viewed as a "covering theory" for the Hamiltonian formulation of field theory (multisymplectic manifolds). This relationship will be discussed in Section 7.4.

Also in [12] it is shown that the Schouten-Nijenhuis brackets of both symmetric and antisymmetric contravariant tensor fields have a natural geometrical interpretation in terms of n-symplectic geometry on the bundle of linear frames LM. Specifically, the restriction of the n-symplectic Poisson bracket to the subspace of GL(n)-tensorial functions is in fact the lift to LM of the Schouten-Nijenhuis brackets. See Section 9.4.

4.5 k-cosymplectic structures and $\mathbb{R}^k \times (T^1_k)^*M$

Let us begin by recalling that a cosymplectic manifold is a triple (M, θ, ω) consisting of a smooth $(2n+1)$-dimensional manifold M with a closed 1-form θ and a closed 2-form ω, such that $\theta \wedge \omega^n \neq 0$. The standard example of a cosymplectic manifold is provided by $(J^1(\mathbb{R}, N) \equiv \mathbb{R} \times T^*N, dt, \pi^*\omega_0)$, with $t : \mathbb{R} \times T^*N \to \mathbb{R}$ and $\pi : \mathbb{R} \times T^*N \to T^*N$ the canonical projections and ω_0 the canonical symplectic form on T^*N.

Definition 4.7 *Let M be a differentiable manifold of dimension $(k+1)n + k$. A family $(\eta_A, \omega_A, V; 1 \leq A \leq k)$, where each η_A is a closed 1-form, each ω_A is a closed 2-form and V is an nk-dimensional integrable distribution on M, such that*

1. $\eta_1 \wedge \cdots \wedge \eta_k \neq 0, \quad \eta_{A\rfloor V} = 0, \quad \omega_{A\rfloor V\times V} = 0,$

2. $(\cap_{A=1}^k \ker \eta_A) \cap (\cap_{A=1}^k \ker \omega_A) = \{0\}, \quad \dim(\cap_{A=1}^k \ker \omega_A) = k,$

*is called a k–*cosymplectic structure *and the manifold M a k–*cosymplectic manifold.

The canonical model for these geometrical structures is $\mathbb{R}^k \times (T^1_k)^*M = J^1(M, \mathbb{R}^k)$. Let $J^1(M, \mathbb{R}^k)$ be the $(k+(k+1)n)$-dimensional manifold of one jets from M to $\mathbb{R}^k$, with elements denoted by $j^1_{x,t}\sigma$. We recall that one jets of mappings from M to $\mathbb{R}^k$ can be identified with the manifold $J^1\pi$ of one jets of sections of the trivial bundle $\pi : \mathbb{R}^k \times M \to M$.

$J^1\pi$ is diffeomorphic to $\mathbb{R}^k \times (T^1_k)^*M$ via the diffeomorphism given by

$$j^1_x\sigma \in J^1\pi \to (\sigma(x), j^1_{x,0}\sigma_x) \in \mathbb{R}^k \times (T^1_k)^*M\,,$$

where $\sigma_x(\tilde{x}) = \sigma(\tilde{x}) - \sigma(x)$ and $\tilde{x}$ denotes an arbitrary point in M.

Let $\tau^* : \mathbb{R}^k \times (T^1_k)^*M \to M$ denote the canonical projection. If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (t^A, x^i, p^A_i), $1 \leq i \leq n$, $1 \leq A \leq k$, on $(\tau^*)^{-1}(U) \equiv \mathbb{R}^k \times (T^1_k)^*U$ are given by

$$t^A(j^1_x\sigma) = t^A, \qquad x^i(j^1_x\sigma) = x^i(x), \qquad p^A_i(j^1_x\sigma) = d(\sigma^A_x)(x)(\frac{\partial}{\partial x^i}_{|x})$$

where $\sigma^A_x = \pi_A \circ \sigma_x$.

An $\mathbb{R}^k$-valued 1-form η_0 and an $\mathbb{R}^k$-valued 2-form ω_0 on $\mathbb{R}^k \times (T^1_k)^*M$ are defined by

$$\eta_0 = \sum_{A=1}^m (\eta_0)_A\, \hat{r}_A = \sum_{A=1}^k ((\pi^1_A)^*dt)\, \hat{r}_A, \quad \omega_0 = \sum_{A=1}^k (\omega_0)_A\, \hat{r}_A = \sum_{A=1}^m (\pi^2_A)^*(\omega_M)\, \hat{r}_A \tag{46}$$

where $\pi^1_A : \mathbb{R}^k \times (T^1_k)^*M \to \mathbb{R}$ and $\pi^2_A : \mathbb{R}^k \times (T^1_k)^*M \to T^*M$ are the projections defined by

$$\pi^1_A((t^B), (p^B)) = t^A\,, \quad \pi^2_A((t^B), (p^B)) = p^A\,,$$

and ω_M is the canonical symplectic form on T^*M. In local coordinates we have

$$(\eta_0)_A = dt^A, \quad (\omega_0)_A = \sum_{i=1}^m dx^i \wedge dp_i^A \quad 1 \leq A \leq k \tag{47}$$

Moreover, let $V = \ker T\mu^*$, where $\mu^* : \mathbb{R}^k \times (T_k^1)^*M \to \mathbb{R}^k \times M$. Then locally

$$V = \langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \rangle \quad 1 \leq A \leq k\,.$$

and the canonical k-cosymplectic structure on $\mathbb{R}^k \times (T_k^1)^*M$ is $((\eta_0)_A, (\omega_0)_A, V)$. Indeed a simple computation in local coordinates shows that the forms $((\eta_0)_A, (\omega_0)_A, V)$ satisfy the conditions of Definition 4.7 .

For any k-cosymplectic structure (η_A, ω_A, V) on M, there exists a family of k vector fields $(\xi_1, \dots, \xi_k)$ characterizated by the conditions

$$\eta_A(\xi_B) = \delta_{AB}, \quad \iota_{\xi_B}\omega_A = 0,$$

for all $1 \leq A, B \leq k$. These vector fields are called the *Reeb vector fields* associated to the k-cosymplectic structure.

Theorem 4.8 *[41] Let $(\eta_A, \omega_A, V, 1 \leq A \leq k)$ be a k-cosymplectic structure on M. About every point of M we can find a local coordinate system (t^A, x^i, p_i^A) such that*

$$(\eta_0)_A = dt^A, \quad (\omega_0)_A = \sum_{i=1}^n dx^i \wedge dp_i^A\,, \quad V = \langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \rangle \quad 1 \leq A \leq k\,,$$

and the Reeb vector fields are given by $\xi_A = \frac{\partial}{\partial t^A}$.

4.6 Multisymplectic structures

In k-symplectic geometry the model is the Whitney sum of k-copies of the cotangent bundle of a manifold M. In multisymplectic geometry [34, 35, 36, 37] one uses a completely different model.

Let E be an m-dimensional differentiable manifold and denote by $\bigwedge^k E$ the bundle of exterior k-forms on E with canonical projection $\rho_k : \bigwedge^k E \to E$. Notice that $\bigwedge^1 E = T^*E$. On $\bigwedge^k E$ there exists a canonical k-form Θ_E defined by

$$(\Theta_E)_\alpha(v_1, \dots, v_k) = \alpha(T\rho_k(v_1), \dots, T\rho_k(v_k))$$

for $\alpha \in \bigwedge^k E$ and $v_1, \ldots, v_k \in T_\alpha(\bigwedge^k E)$. This is a direct extension of the construction of the canonical Liouville 1-form on a cotangent bundle.

Next, we define a $(k+1)$-form Ω_E by

$$\Omega_E = -d\Theta_E .$$

Taking bundle coordinates $(x^i, p_{i_1 \ldots i_k})$, $1 \leq i \leq m$, $1 \leq i_1 < \cdots < i_k \leq m$, on $\bigwedge^k E$, we have

$$\Theta_E = p_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} , \quad \Omega_E = -dp_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Assume that E itself is fibered over some manifold M, with projection $\pi : E \to M$. For any r, with $0 \leq r \leq k$, let $\bigwedge_r^k E$ denote the bundle over E consisting of those exterior k-forms on E which vanish whenever r of its arguments are vertical tangent vectors with respect to π. Obviously, $\bigwedge_r^k E$ is a vector subbundle of $\bigwedge^k E$, and we will denote by $i_{k,r} : \bigwedge_r^k E \to \bigwedge^k E$ the natural inclusion.

The restriction of Θ_E and Ω_E to $\bigwedge_r^k E$ will be denoted by Θ_E^r and Ω_E^r, respectively; that is

$$\Theta_E^r = i_{k,r}^* \Theta_E, \qquad \Omega_E^r = i_{k,r}^* \Omega_E ,$$

and, clearly, $\Omega_E^r = -d\Theta_E^r$.

Based on the properties of the $(k+1)$-forms Ω_E and Ω_E^r, we introduce the following definition.

Definition 4.9 *A closed $(k+1)$-form α on a manifold N is called* multisymplectic *if it is non-degenerate in the sense that for a tangent vector X on N, $X \,\lrcorner\, \alpha = 0$ if and only if $X = 0$. The pair (N, α) will then called a* multisymplectic manifold.

Of course the manifolds $(\bigwedge^k E, \Omega_E)$ and $(\bigwedge_r^k E, \Omega_E^r)$, $0 \leq r \leq k$, are multisymplectic.

To develop the multisymplectic formalism of field theory we will use the canonical multisymplectic manifold $(\bigwedge_2^n E, \Omega_E^2)$ and the manifold $(\bigwedge_1^n E, \Omega_E^1)$. If M is oriented with volume form ω we can consider coordinates (x^i, y^A) on E such that $\omega = d^n x = dx^1 \wedge \cdots \wedge dx^n$. Elements of $\bigwedge_1^n E$ and $\bigwedge_2^n E$ can be written, respectively, as follows

$$p\, d^n x, \qquad p\, d^n x + p_A^i dy^A \wedge d^{n-1} x_i$$

where $d^{n-1}x_i = \frac{\partial}{\partial x^i} \lrcorner\, d^n x$. Then we take local coordinates (x^i, y^A, p) on $\bigwedge_1^n E$ and (x^i, y^A, p, p_A^i) on $\bigwedge_2^n E$. Therefore the canonical multisymplectic $(n+1)$-form Ω_E^2 on $\bigwedge_2^n E$ is locally given by

$$\Omega_E^2 = -dp \wedge d^n x - dp_A^i \wedge dy^A \wedge d^{n-1}x_i \tag{48}$$

and $\Theta_E^2 = p\, d^n x + p_A^i dy^A \wedge d^{n-1}x_i$.

Remark In [38] the authors have developed a geometrical study of multisymplectic manifolds, exhibiting the complexity of a classification. A characterization of multisymplectic manifolds which are exterior bundles can be found in [39].

5 Relationships among the cotangent-like structures

Here we show how the symplectic, k-symplectic, m-symplectic and similar structures are related. We also venture further into the realm of multisymplectic geometry by showing how the canonical k-symplectic structure is induced from a special case of the multisymplectic structure on $J^1\pi^*$. We use here the definition of $J^1\pi^*$ given in [40] rather than the affine dual definition of $J^1\pi^*$ given in [34].

5.1 Relationships among T^*M, $(T_k^1)^*M$, and LM

In Section 4.2 we have already seen the relationship between the k-symplectic structure on $(T_k^1)^*M$ and the symplectic structure on T^*M. The relationship between the canonical symplectic structure on T^*M and the soldering form on LM can be found in [11]: if θ_0 is the Liouville 1-form on T^*M and θ the soldering 1-form on LM then

$$(\theta_0)_{[u,\alpha]}(\bar{X}_{[u,\alpha]}) \;=\; \alpha(\theta_u(X_u)), \quad [u,s] \in T^*M \equiv LM \times_{Gl(n,\mathbb{R})} (\mathbb{R}^n)^*\,.$$

In this equation u is a point in LM, $[u, \alpha]$ denotes a point (equivalence class) in T^*M thought of as the associated bundle $LM \times_{GL(m,\mathbb{R})} (\mathbb{R}^n)^*$ and

$$\bar{X}_{[u,\alpha]} \in T_{[u,\alpha]}T^*M\,, \quad X_u \in T_u(LM)$$

are vectors that project to the same vector on M, and $\alpha \in \mathbb{R}^{n*}$ is non-zero.

5.2 The multisymplectic form and the canonical k-symplectic structure

Now we shall describe the relationship between the canonical multisymplectic form Ω^2_E on $\bigwedge^k_2 E$ and the canonical k-symplectic structure on $(T^1_k)^*M$ when E is the trivial bundle $E = \mathbb{R}^k \times M \to \mathbb{R}^k$. In this case $\bigwedge^k_2 E$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*M$. Let us recall that $\bigwedge^k_2 E$ is the vector bundle

$$\Lambda^k_2(\mathbb{R}^k \times M) = \{\alpha_{(t,x)} \in \Lambda^k(\mathbb{R}^k \times M) \,:\, v \lrcorner\; w \lrcorner\; \alpha_{(t,x)} = 0 \,\forall v, w \in (V\pi)_{(t,x)}\}$$

where $V\pi$ is the vertical fiber bundle corresponding to π.

We define

$$\begin{array}{rccc} \Psi: & \Lambda^k_2(\mathbb{R}^k \times M) & \longrightarrow & \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*M \\ & \alpha_{(t,x)} & \to & (t, r, (\alpha^1)_x, \dots, (\alpha^k)_x) \end{array}$$

where

$$r = \alpha_{(t,x)}(\frac{\partial}{\partial t^1}(t,x), \dots, \frac{\partial}{\partial t^k}(t,x))$$

and

$$(\alpha^B)_x(-) = i^*_t(\alpha_{(t,x)}(\frac{\partial}{\partial t^1}(t,x), \dots, \frac{\partial}{\partial t^{B-1}}(t,x), -, \frac{\partial}{\partial t^{B+1}}(t,x), \dots, \frac{\partial}{\partial t^k}(t,x))) \quad 1 \leq B \leq k,$$

where $i_t : M \to R^k \times M$ denotes the inclusion $x \to (t,x)$.

The inverse of Ψ

$$\begin{array}{rccc} \Psi^{-1}: & \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*M & \longrightarrow & \Lambda^k_2(\mathbb{R}^k \times M) \\ & (t, r, (\alpha^1)_x, \dots, (\alpha^k)_x) & \mapsto & \alpha_{(t,x)} \end{array}$$

is given by

$$\alpha_{(t,x)} = r(d^k t)_{(t,x)} + (pr_2^*)_{(t,x)}((\alpha^B)_x) \wedge (d^{k-1}t^B)_{(t,x)}$$

where

$$d^k t = dt^1 \wedge \dots \wedge dt^k, \qquad d^{k-1}t^B = \frac{\partial}{\partial t^B} \lrcorner\; dt^k$$

and $pr_2 : \mathbb{R}^k \times M \to M$ is the canonical projection.

Elements of $\bigwedge^k_2 E$ can be written uniquely as

$$p^B_i \, dx^i \wedge d^{k-1}t^B \, + \, p \, d^k t$$

where (x^i) are coordinates on M. Let us denote by (t_B, p, x^i, p_i^A) the corresponding coordinates on $\bigwedge_2^k E \equiv \mathbb{R}^k \times \mathbb{R} \times (T_k^1)^*M$. Locally Ψ is written as the identity.

The canonical k-form on $\bigwedge_2^k E \equiv \mathbb{R}^k \times \mathbb{R} \times (T_k^1)^*M$ is locally given in this case by

$$\Theta_E^2 = p_i^B\, dx^i \wedge d^{k-1}t^B \;+\; p\, d^k t \tag{49}$$

and the corresponding canonical multisymplectic $(k+1)$-form $\Omega_E^2 = -d\Theta_E^2$ is locally given by

$$\Omega_E^2 = dx^i \wedge dp_i^B \wedge d^{k-1}t^B - dp \wedge d^k t$$

Let $i : (T_k^1)^*M \to \mathbb{R}^k \times \mathbb{R} \times (T_k^1)^*M$ be the natural inclusion. We define on $(T_k^1)^*M$ the 1-forms λ_B, $1 \leq B \leq k$, by

$$\lambda_B(-) = i^*(\Theta_E^2(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{B-1}}, -, \frac{\partial}{\partial t^{B+1}}, \ldots, \frac{\partial}{\partial t^k}),$$

and from (49) we deduce $\lambda_B = p_i^B\, dx^i$. Hence λ_B is the Liouville form on the B-th copy T^*M of $(T_k^1)^*M$. To get this local expression apply λ_B to the partials $\partial/\partial x^i$ and $\partial/\partial p_i^B$.

Therefore the 2 forms

$$\omega_B = -d\lambda_B = dx^i \wedge dp_i^B, \quad 1 \leq B \leq k$$

define the canonical k-symplectic structure on $(T_k^1)^*M$, and ω_B can also be defined as follows

$$\omega_B(-,-)) = i^*(\Omega_E^2(-, \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{B-1}}, -, \frac{\partial}{\partial t^{B+1}}, \ldots, \frac{\partial}{\partial t^k}), \tag{50}$$

The case $k = 1$ gives us the canonical symplectic structure of T^*M.

Proposition 5.1 *The relationship between the 2-forms of the canonical k-symplectic structure on $(T_k^1)^*M$ and the canonical multisymplectic form Ω_E^2 is given by (50).* ■

6 Field Theory on k-symplectic and k-cosymplectic manifolds

Here we discuss the polysymplectic formalism [4] for Hamiltonian and Lagrangian field theory using k-symplectic manifolds. We discuss the Günther's formalism (autonomous case) using the k-symplectic structures and the k-tangent structures. The non autonomous case will be developed using the k-cosymplectic structures and the stable k-tangent structures [33, 41].

6.1 k-vector fields

Let M be an arbitrary manifold and $\tau : T^1_k M \longrightarrow M$ its k-tangent bundle .

Definition 6.1 *A section* $\mathbf{X} : M \longrightarrow T^1_k M$ *of the projection* τ *will be called a* k-vector field *on* M*.*

Since $T^1_k M$ can be canonically identified with the Whitney sum $T^1_k M \equiv TM \oplus \cdots \oplus TM$ of k copies of TM, we deduce that a k-vector field $\mathbf{X}$ defines a family of vector fields $X_1, \dots, X_k$ on M.

Definition 6.2 *An* integral section *of the k-vector field* $\mathbf{X}$ *on* M *is a map* $\phi : U \subset \mathbb{R}^k \to M$*, where* U *is an open subset of* $\mathbb{R}^k$ *such that*

$$\phi_*(t)(\frac{\partial}{\partial t^A}) = X_A(\phi(t)) \quad \forall t \in U, \quad 1 \leq A \leq k,$$

or equivalently, ϕ *satisfies*

$$X \circ \phi = \phi^{(1)}, \tag{51}$$

where $\phi^{(1)}$ *is the first prolongation of* ϕ *defined by*

$$\begin{array}{rccl} \phi^{(1)} : & U \subset \mathbb{R}^k & \longrightarrow & T^1_k M \\ & t & \longrightarrow & \phi^{(1)}(t) = j^1_0 \phi_t \end{array}$$

where $\phi_t(s) = \phi(s+t)$ *for all* $t, s \in \mathbb{R}$*. If* $\mathbf{X}$ *has an integral section,* $\mathbf{X}$ *is said to be* integrable.

Remark Let us consider the trivial bundle $\pi : E = R^k \times M \to R^k$. A jet field γ on π (see [17]) is a section of the projection $\pi_{1,0} : J^1\pi \equiv \mathbb{R}^k \times T^1_k M \longrightarrow E \equiv \mathbb{R}^k \times M$. If we identify each k-vector field $\mathbf{X}$ on M with the jet field $\gamma = (id_{\mathbb{R}^k}, X)$, that is $\gamma(t,x) = (t, X_1(x), \dots, X_k(x))$, then the integral sections of the jet field γ correspond, as defined by Günther, to the *solutions* of the k-vector field $\mathbf{X}$.

We remark that if ϕ is an integral section of a k-vector field $(X_1, \dots, X_k)$ then each curve on M defined by $\phi_A = \phi \circ h_A$, where $h_A : \mathbb{R}^n \to \mathbb{R}^k$ is the natural inclusion $h_A(t) = (0, \dots, t, \dots, 0)$, is an integral curve of the vector field X_A on M, with $1 \leq A \leq k$. We refer to [42, 43] for a discussion on the existence of integral sections.

6.2 Hamiltonian formalism and k-symplectic structures

In this section, following the ideas of Günther [4], we will describe the Hamilton equations, for an autonomous Hamiltonian, in terms of the geometry of k-symplectic structures, showing that the role played by symplectic manifolds in classical mechanics is here played by the k-symplectic manifolds.

Let $(M, \omega_A, V; 1 \leq A \leq k)$ be a k–symplectic manifold. Since M is a polysymplectic manifold let us consider the vector bundle morphism defined by Günther:

$$\begin{array}{rccl} \Omega^\sharp : & T^1_k M & \longrightarrow & T^*M \\ & (X_1, \ldots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \ldots, X_k) = \displaystyle\sum_{A=1}^k X_A \lrcorner\, \omega_A \,. \end{array} \tag{52}$$

Definition 6.3 *Let $H : M \longrightarrow \mathbb{R}$ be a function on M. Any k-vector field $(X_1, \ldots, X_k)$ on M such that*

$$\Omega^\sharp(X_1, \ldots, X_k) = dH$$

will be called an evolution k-vector field *on M associated with the Hamiltonian function H.*

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [41] that there always exists an *evolution k-vector field* associated with a Hamiltonian function H.

Let (x^i, p^A_i) be a local coordinate system on M. Then we have

Proposition 6.4 *If $(X_1, \ldots, X_k)$ is an integrable evolution k-vector field associated to H then its integral sections*

$$\begin{array}{rccl} \sigma : & \mathbb{R}^k & \longrightarrow & M \\ & (t^B) & \longrightarrow & (\sigma^i(t^B), \sigma^A_i(t^B)), \end{array}$$

are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [44]:

$$\frac{\partial H}{\partial x^i} = -\sum_{A=1}^k \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n,\, 1 \leq \; A \leq k \,.$$

6.3 Hamiltonian formalism and k-cosymplectic structures

In this section we will describe the Hamilton equations for a non-autonomous Hamiltonian in terms of the geometry of k-cosymplectic structures, showing that the role played by cosymplectic manifolds in classical mechanics (see [45, 46, 47]) is here played by the k-cosymplectic manifolds.

Let $(M, \eta_A, \omega_A, V; 1 \leq A \leq k)$ be a k–cosymplectic manifold. Let us consider the vector bundle morphism defined by :

$$\begin{array}{rccl} \Omega^\sharp : & T^1_k M & \longrightarrow & T^*M \\ & (X_1, \ldots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \ldots, X_k) = \displaystyle\sum_{A=1}^{k} X_A \lrcorner\, \omega_A + \eta_A(X_A)\eta_A \, . \end{array} \tag{53}$$

Let ξ_A the Reeb vector fields associated to the k-cosymplectic structure (η_A, ω_A, V). Notice here that the hamiltonian $H(t^A, x^i, p^A_i)$ is non-autonomous.

Definition 6.5 *Let $H : M \longrightarrow \mathbb{R}$ be a function on M. Any k-vector field $(X_1, \ldots, X_k)$ on M such that*

$$\eta_A(X_B) = \delta^A_B, \quad \Omega^\sharp(X_1, \ldots, X_k) = dH + \sum_{iA=1}^{k} (1 - \xi_A(H))\eta_A$$

will be called an evolution k-vector field *on M associated with the Hamiltonian function H for all $1 \leq A, B \leq k$.*

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [41] that there always exists an evolution k-vector field associated with a Hamiltonian function H.

Let (t^A, x^i, p^A_i) be a local coordinate system on M. Then we have

Proposition 6.6 *If $(X_1, \ldots, X_k)$ is an integrable evolution k-vector field associated to H then its integral sections*

$$\begin{array}{rccl} \sigma : & \mathbb{R}^k & \longrightarrow & M \\ & (t^B) & \longrightarrow & (\sigma^A(t^B), \sigma^i(t^B), \sigma^A_i(t^B)), \end{array}$$

satisfy $\sigma^A(t^1, \ldots, t^k) = t^A$ and are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [44]:

$$\frac{\partial H}{\partial x^i} = -\sum_{A=1}^{k} \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n, 1 \leq A \leq k \, . \quad \blacksquare$$

6.4 Second Order Partial Differential Equations on $T^1_k M$

The idea of this subsection is to characterize the integrable k-vector fields on T^1_kM such that their integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to M.

Definition 6.7 *A k-vector field on T^1_kM, that is, a section $\xi \equiv (\xi_1, \dots, \xi_k) : T^1_kM \to T^1_k(T^1_kM)$ of the projection $\tau_{T^1_kM} : T^1_k(T^1_kM) \to T^1_kM$, is a* Second Order Partial Differential Equation (SOPDE) *if and only if it is also a section of the vector bundle $T^1_k\tau_M : T^1_k(T^1_kM) \to T^1_kM$, where $T^1_k(\tau_M)$ is defined by $T^1_k(\tau_M)(j^1_0\sigma) = j^1_0(\tau_M \circ \sigma)$.*

Let (x^i) be a coordinate system on M and (x^i, v^i_A) the induced coordinate system on T^1_kM. From the definition we deduce that the local expression of a SOPDE ξ is

$$\xi_A(x^i, v^i_A) = v^i_A \frac{\partial}{\partial x^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k. \tag{54}$$

We recall that the first prolongation $\phi^{(1)}$ of $\phi : U \subset \mathbb{R}^k \to M$ is defined by

$$\begin{array}{rccl} \phi^{(1)}: & U \subset \mathbb{R}^k & \longrightarrow & T^1_kM) \\ & t & \longrightarrow & \phi^{(1)}(t) = j^1_0\phi_t \end{array}$$

where $\phi_t(s) = \phi(s + t)$ for all $t, s \in \mathbb{R}$. In local coordinates:

$$\phi^{(1)}(t^1, \dots, t^k) = (\phi^i(t^1, \dots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \dots, t^k)), \qquad 1 \leq A \leq k\,,\, 1 \leq i \leq n\,. \tag{55}$$

Proposition 6.8 *Let ξ an integrable k-vector field on T^1_kM. The necessary and sufficient condition for ξ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is*

$$\xi_A(\phi^{(1)}(t)) = \phi^{(1)}_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \dots, k$. These maps ϕ will be called solutions of the SOPDE ξ.

¿From (55) and (54) we have

Proposition 6.9 *$\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE $\xi = (\xi_1, \dots, \xi_k)$, locally given by (54), if and only if*

$$\frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{(1)}(t)), \qquad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\zeta_A)^i_B(\phi^{(1)}(t)).$$

If $\xi : T^1_k M \to T^1_k T^1_k M$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to T^1_k M$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$ where $\tau_M : T^1_k M \to M$ is the canonical projection.

Now we show how to characterize the SOPDEs using the canonical k-tangent structure of $T^1_k M$.

Definition 6.10 *The* canonical vector field C *on* $T^1_k M$ *is the infinitesimal generator of the one parameter group*

$$\begin{array}{ccc} \mathbb{R} \times (T^1_k M) & \longrightarrow & T^1_k M \\ (s, (x^i, v^i_B)) & \longrightarrow & (x^i, e^s\, v^i_B)\,. \end{array}$$

Thus C *is locally expressed as follows:*

$$C = \sum_B C_B = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B}, \tag{56}$$

where each C_B *corresponds with the canonical vector field on the* B*-th copy of* TM *on* $T^1_k M$.

Let us remark that each vector field C_A on $T^1_k M$ can also be defined using the A-lifts of vectors as follows: $C_A((v_1)_q, \ldots, (v_k)_q) = ((v_A)_q)^A(v))\,.$

From (5), (54) and (56) we deduce the following

Proposition 6.11 *A* k*-vector field* $\xi = (\xi_1, \ldots, \xi_k)$ *on* $T^1_k M$ *is a SOPDE if and only if*

$$J^A(\xi_A) = C_A, \qquad \forall\, 1 \leq A \leq k,$$

where $(J^1, \ldots, J^k)$ *is the canonical* k*-tangent structure on* $T^1_k M$.

6.5 Lagrangian formalism and k-tangent structures

Given a Lagrangian function of the form $L = L(x^i, v^i_A)$ one obtains, by using a variational principle, the *generalized Euler-Lagrange equations* for L:

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left(\frac{\partial L}{\partial v^i_A}\right) - \frac{\partial L}{\partial x^i} = 0, \qquad v^i_A = \frac{\partial x^i}{\partial t^A}. \tag{57}$$

In this section, following the ideas of Günther [4], we will describe the above equations (57) in terms of the geometry of k-tangent structures. In classical mechanics the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play complementary roles [21, 22, 23, 24, 25]. Our purpose in this section is to show that the k-symplectic structures and the k-tangent structures play similarly complementary roles.

First of all, we note that such a L can be considered as a function $L : T^1_k M \to \mathbb{R}$ with M a manifold with local coordinates (x^i). Next, we construct a k–symplectic structure on the manifold $T^1_k M$, using its canonical k–tangent structure for each $1 \leq A \leq k$. We consider:

- the vertical derivation $\imath_{J^A}$ of type $\imath_*$ defined by J^A, which is defined by

$$\iota_{J^A} f = 0$$
$$(\iota_{J^A}\alpha)(X_1, \ldots, X_p) = \sum_{j=1}^{p} \alpha(X_1, \ldots, J^A X_j, \ldots, X_p)\,,$$

  for any function f and any p-form α on $T^1_k M$;

- the vertical differentation d_{J^A} of forms on $T^1_k M$ defined by

$$d_{J^A} = [\imath_{J^A}, d] = \imath_{J^A} \circ d - d \circ \imath_{J^A}\,,$$

  where d denotes the usual exterior differentation.

Let us consider the 1–forms $(\beta_L)_A = d_{J^A} L\,,\ 1 \leq A \leq k$. In a local coordinate system (x^i, v^i_A) we have

$$(\beta_L)_A = \frac{\partial L}{\partial v^i_A} dx^i,\ 1 \leq A \leq k. \tag{58}$$

Definition 6.12 *A Lagrangian L is called* regular *if and only if*

$$det(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}) \neq 0, \qquad 1 \leq i, j, \leq n, \quad 1 \leq A, B \leq k\,. \tag{59}$$

By introducing the following 2–forms $(\omega_L)_A = -d(\beta_L)_A\,,\ 1 \leq A \leq k$, one can easily prove the following.

Proposition 6.13 *$L : T^1_k M \longrightarrow \mathbb{R}$ is a regular Lagrangian if and only if $((\omega_L)_1, \ldots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$, where V denotes the vertical distribution of $\tau : T^1_k M \to M$.* ■

Let $L : T^1_k M \longrightarrow \mathbb{R}$ be a regular Lagrangian and let us consider the k–symplectic structure $((\omega_L)_1, \ldots, (\omega_L)_k, V)$ on $T^1_k M$ defined by L. Let $\Omega^\sharp_L$ be the morphism defined by this k–symplectic structure

$$\Omega^\sharp_L : T^1_k(T^1_k M) \longrightarrow T^*(T^1_k M).$$

Thus, we can set the following equation:

$$\Omega^\sharp_L(X_1, \ldots, X_k) = dE_L, \tag{60}$$

where $E_L = C(L) - L$, and where C is the canonical vector field of the vector bundle $\tau : T^1_k M \to M$.

Proposition 6.14 *Let L be a regular Lagrangian. If $\xi = (\xi_1, \cdots, \xi_k)$ is a solution of (60) then it is a SOPDE. In addition if ξ is integrable then the solutions of ξ are solutions of the Euler-Lagrange equations (57).*

Proof It is a direct computation in local coordinates using (54), (56) , (58) and (59). ■.

Remark The *Legendre map* defined by Günther [4]

$$FL : T^1_k M \longrightarrow (T^1_k)^* M$$

can be described here as follows: if $v_x = (v_1, \ldots, v_k)_x \in (T^1_k M)_q$ with $q \in M$ and $v_A \in T_q M$, then $\mathcal{FL}(v_x) = (\tilde{v^1}, \ldots, \tilde{v}^k) \in (T^1_k M)^*_x$, where $\tilde{v}^A \in T^*_x M$ is given by

$$\tilde{v}^A(z) = (\beta_L)_A(\bar{z}), \quad 1 \leq A \leq k,$$

for any $z \in T_x M$, where $\bar{z} \in T_{v_x}(T^1_k M)$ with $\tau_*(\bar{z}) = z$.

From (58) we deduce that FL is locally given by

$$(x^i, v^i_A) \longrightarrow (x^i, \frac{\partial L}{\partial v^i_A}). \tag{61}$$

and from (58) and (61) we deduce the following

Lemma 6.15 *For every $1 \leq A \leq k$, we have $(\omega_L)_A = FL^*\omega_A$, where $\omega_1, \ldots, \omega_k$ are the 2-forms of the canonical k–symplectic structure of $(T^1_k)^* M$.* ■

Then from (65) we get that

Proposition 6.16 *Let L be a Lagrangian. The following conditions are equivalent:*

1) L is regular.

2) FL is a local diffeomorphism.

3) $((\omega_L)_1, \dots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$. ■

6.6 Second order partial differential equations on $\mathbb{R}^k \times T^1_k M$

The idea of this subsection is to characterize the integrable k-vector fields on $\mathbb{R}^k \times T^1_k M$ such that their integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to M.

Definition 6.17 *A k-vector field on $\mathbb{R}^k \times T^1_k M$, that is, a section $\xi \equiv (\xi_1, \dots, \xi_k) : \mathbb{R}^k \times T^1_k M \to T^1_k(\mathbb{R}^k \times T^1_k M)$ of the projection $\tau_{\mathbb{R}^k \times T^1_k M} : T^1_k(\mathbb{R}^k \times T^1_k M) \to \mathbb{R}^k \times T^1_k M$, is a* Second Order Partial Differential Equation (SOPDE) *if and only if:*

1) $dt^A(\xi_B) = \delta^A_B$

2) $Tpr_2 \circ \xi_B \circ i_t$ is a SOPDE on $T^1_k M$, $\forall t \in \mathbb{R}^k$, where $pr_2 : \mathbb{R}^k \times T^1_k M \to T^1_k M$ is the canonical projection and $i_t : T^1_k M \to \mathbb{R}^k \times T^1_k M$ is the canonical inclusion.

Let (x^i) be a coordinate system on M and (t^A, x^i, v^i_A) the induced coordinate system on $\mathbb{R}^k \times T^1_k M$. From (63) we deduce that the local expression of a SOPDE ξ is

$$\xi_A(x^i, v^i_A) = \frac{\partial}{\partial t_A} + v^i_A \frac{\partial}{\partial x^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k \tag{62}$$

where $(\xi_A)^i_B$ are functions on $\mathbb{R}^k \times T^1_k M$.

Definition 6.18 *For $\phi : \mathbb{R}^k \to M$ a map, we define the* first prolongation *$\phi^{(1)}$ of ϕ as the map*

$$\begin{array}{rcccc} \phi^{(1)} : \mathbb{R}^k & \longrightarrow & J^1\pi & \equiv & \mathbb{R}^k \times T^1_k M, \\ t & \longrightarrow & j^1_t \phi & \equiv & (t, j^1_0 \phi_t) \end{array}$$

In local coordinates:

$$\phi^{(1)}(t^1, \dots, t^k) = (t^1, \dots, t^k, \phi^i(t^1, \dots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \dots, t^k)), \qquad 1 \leq A \leq k\,,\, 1 \leq i \leq n\,. \tag{63}$$

Proposition 6.19 *Let ξ an integrable k-vector field on $\mathbb{R}^k \times T^1_k M$. The necessary and sufficient condition for ξ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is*

$$\xi_A(\phi^{(1)}(t)) = \phi^{(1)}_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \ldots, k$.

These maps ϕ will be called solutions *of the SOPDE ξ.*

¿From (63) and (62) we have

Proposition 6.20 *$\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE ξ, locally given by (62), if and only if*

$$\frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{(1)}(t)), \qquad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\zeta_A)^i_B(\phi^{(1)}(t)).$$

If ξ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to \mathbb{R}^k \times T^1_k M$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$ where $\tau_M : \mathbb{R}^k \times T^1_k M \to M$ is the canonical projection. Now we show how to characterize the SOPDEs on $\mathbb{R}^k \times T^1_k M$ using the canonical k-tangent structure of $T^1_k M$. Let us consider on $\mathbb{R}^k \times T^1_k M$ the tensor fields $\hat{J}^1, \ldots, \hat{J}^k$ of type $(1,1)$, defined as follows:

$$\hat{J}^A = J^A - C_A \otimes dt^A, \quad 1 \leq A \leq k\,.$$

where we have transported the canonical k-tangent structure $(J^1, \ldots, J^k)$ of $T^1_k M$ to $\mathbb{R}^k \times T^1_k M$.

Proposition 6.21 *A k-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $\mathbb{R}^k \times T^1_k M$ is a SOPDE if and only if*

$$\hat{J}^A(\xi_A) = 0, \qquad \bar{\eta}_A(\xi_B) = \delta_{AB},$$

for all $1 \leq A, B \leq k$.

Remark: Let us consider the trivial bundles $\pi : E = \mathbb{R}^k \times M \to \mathbb{R}^k$ and $\pi_1 : \mathbb{R}^k \times T^1_k M \to \mathbb{R}^k$. We identify each SOPDE $(\xi_1, \ldots, \xi_k)$ with the following *semi-holonomic second order jet field*

$$\begin{array}{rcl} J^1\pi \equiv \mathbb{R}^k \times T^1_k M & \to & J^1\pi_1 \equiv \mathbb{R}^k \times T^1_k(T^1_k M) \\ (t^A, q^i, v^i_A) & \to & (t^A, q^i, v^i_A, v^i_A, (\xi_A)^i_B) \end{array}$$

If the SOPDE ξ on $\mathbb{R}^k \times T^1_k M$ is integrable, then its integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to M and then ξ defines a second-order jet field Γ on π whose coordinate representation of the corresponding connection $\tilde{\Gamma}$ is

$$\tilde{\Gamma} = dt^A \otimes \left(\frac{\partial}{\partial t^A} + v^i_A \frac{\partial}{\partial q^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B} \right) ,$$

since $(\xi_A)^i_B = (\xi_B)^i_A$ (see [17]).

The integrability of the SOPDE is equivalent to the condition given by $\mathcal{R} = 0$, where $\mathcal{R}$ is the curvature tensor of the above connection (see [42] and [17]).

6.7 Lagrangian formalism and stable k-tangent structures

Given a nonautonomous Lagrangian $\mathcal{L} = \mathcal{L}(t^A, q^i, v^i_A)$ one realizes that such an $\mathcal{L}$ can be considered as a function $\mathcal{L} : \mathbb{R}^k \times T^1_k M \to \mathbb{R}$.

In this section we shall give a geometrical description of Euler Lagrange equations (57) using a k-cosymplectic structure on $\mathbb{R}^k \times T^1_k M$ associated to the regular Lagrangian $\mathcal{L}$. This k-cosymplectic structure shall be constructed using the *canonical* tensor fields $\tilde{J}^A$, $1 \leq A \leq k$ of type $(1, 1)$ on $\mathbb{R}^k \times T^1_k M$ defined by

$$\tilde{J}^A = \frac{\partial}{\partial t^A} \otimes dt^A + J^A = \frac{\partial}{\partial t^A} \otimes dt^A + \sum_{i=1}^{n} \frac{\partial}{\partial v^i_A} \otimes dq^i \,, \qquad 1 \leq A \leq k \,,$$

where we have transported the canonical k-tangent structure $(J^1, \dots, J^k)$ of $T^1_k M$ to $\mathbb{R}^k \times T^1_k M$. The family $(\tilde{J}^A, dt^A, \frac{\partial}{\partial t_A})$ is called the *canonical stable k-tangent structure* on $\mathbb{R}^k \times T^1_k M$.

For each $1 \leq A \leq k$, we define:

- the vertical derivation $\imath_{J^A}$ of forms on $\mathbb{R}^k \times T^1_k M$ by

$$\imath_{\tilde{J}^A} f = 0 \,, \quad (\imath_{\tilde{J}^A}\alpha)(X_1, \dots, X_p) = \sum_{j=1}^{p} \alpha(X_1, \dots, \tilde{J}^A X_j, \dots, X_p) \,,$$

  for any function f and any p-form α on $\mathbb{R}^k \times T^1_k M$;

- the vertical differentation $d_{\tilde{J}^A}$ of forms on $\mathbb{R}^k \times T^1_k M$ by

$$d_{\tilde{J}^A} = [\imath_{\tilde{J}^A}, d] = \imath_{\tilde{J}^A} \circ d - d \circ \imath_{\tilde{J}^A} \,,$$

  where d denotes the usual exterior differentation.

Let us consider the 1–forms

$$(\beta_{\mathcal{L}})_A = d_{\tilde{J}^A}\mathcal{L} - \xi_A(\mathcal{L})dt^A, \quad 1 \leq A \leq k\,.$$

In bundle coordinates (t^A, q^i, v^i_A) we have

$$(\beta_{\mathcal{L}})_A = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial v^i_A} dq^i,\ 1 \leq A \leq k\,. \tag{64}$$

Definition 6.22 *A Lagrangian $\mathcal{L}$ is called* regular *if and only if the Hessian matrix*

$$\left(\frac{\partial^2 \mathcal{L}}{\partial v^i_A \partial v^j_B} \right) \tag{65}$$

is non–singular.

Now, we introduce the following 2–forms

$$(\omega_{\mathcal{L}})_A = -d(\beta_{\mathcal{L}})_A\,,\ 1 \leq A \leq k\,.$$

Using local coordinates one can easily prove the following proposition.

Proposition 6.23 *Let $\mathcal{L} : \mathbb{R}^k \times T^1_k M \longrightarrow R$ be a regular Lagrangian, and $V_{1,0}$ the vertical distribution of the bundle $\pi_{1,0} : \mathbb{R}^k \times T^1_k M \longrightarrow \mathbb{R}^k \times M$. Then, $\mathcal{L}$ is regular if and only if $(\mathbb{R}^k \times T^1_k M, \bar{\eta}_A, (\omega_{\mathcal{L}})_A, V_{1,0})$ is a k–cosymplectic manifold.* ■

Let $\mathcal{L} : \mathbb{R}^k \times T^1_k M \longrightarrow \mathbb{R}$ be a regular Lagrangian and $(dt^A, (\omega_L)_A, V_{1,0})$ the associated k-cosymplectic structure on $\mathbb{R}^k \times T^1_k M$. The equations

$$dt^A((\xi_{\mathcal{L}})_B) = \delta^A_B, \quad (\xi_{\mathcal{L}})_A \lrcorner\ (\omega_{\mathcal{L}})_B = 0, \qquad 1 \leq A, B \leq k\,. \tag{66}$$

define the Reeb vector fields $\{(\xi_{\mathcal{L}})_1, \ldots, (\xi_{\mathcal{L}})_k\}$ on $\mathbb{R}^k \times T^1_k M$ which are locally given by

$$(\xi_{\mathcal{L}})_A = \frac{\partial}{\partial t^A} + ((\xi_L)_A)^i_B \frac{\partial}{\partial v^i_B}\,, \tag{67}$$

where the functions $((\xi_{\mathcal{L}})_A)^i_B$ satisfy

$$\frac{\partial^2 \mathcal{L}}{\partial t^A \partial v^j_C} + \frac{\partial^2 \mathcal{L}}{\partial v^i_B \partial v^j_C} ((\xi_{\mathcal{L}})_A)^i_B = 0\,, \tag{68}$$

for all $1 \leq A, B, C \leq k$ and $1 \leq i, j \leq n$.

Since $\mathcal{L}$ is regular, from the local conditions (68) we can define, in a neighbourhood of each point of $\mathbb{R}^k \times T^1_k M$, a k–vector field that satisfies (66). Next one can construct a global k–vector field ξ_L, which is a solution of (66), by using a partition of unity.

Let $\mathcal{L}$ be a regular Lagrangian and let $\Omega^\sharp_\mathcal{L}$ be the $\sharp$-morphism defined by the k-cosymplectic structure $(dt^A, (\omega_\mathcal{L})_A, V_{1,0})$, as in (52):

$$\begin{array}{rccl} \Omega^\sharp_\mathcal{L}: & T^1_k(\mathbb{R}^k \times T^1_k M) & \longrightarrow & T^*(\mathbb{R}^k \times T^1_k M) \\ & (X_1, \ldots, X_k) & \longrightarrow & \Omega^\sharp_\mathcal{L}(X_1, \ldots, X_k) = \displaystyle\sum_{A=1}^k X_A \rfloor\, (\omega_\mathcal{L})_A + dt^A(X_A) dt^A \,. \end{array} \tag{69}$$

A direct computation in local coordinates proves the following Proposition.

Proposition 6.24 *Let $\mathcal{L}$ be a regular Lagrangian and let $X = (X_1, \ldots, X_k)$ be a k-vector field such that*

$$\begin{array}{l} dt^A(X_B) = \delta_{AB}, \quad 1 \leq A, B \leq k \\ \Omega^\sharp_\mathcal{L}(X_1, \ldots, X_k) = dE_\mathcal{L} + \displaystyle\sum_{A=1}^k (1 - (\xi_\mathcal{L})_A(E_\mathcal{L})) dt^A \end{array} \tag{70}$$

where $E_\mathcal{L} = C(\mathcal{L}) - \mathcal{L}$. Then $X = (X_1, \ldots, X_k)$ is a SOPDE. In addition, if $X = (X_1, \ldots, X_k)$ is integrable then its solutions satisfy the Euler-Lagrange equations (57).

In conclussion, we can consider Eqs. (70) as a *geometric version* of the Euler-Lagrange field equations for a regular Lagrangian.

Remark We have given a geometric version of the Euler-Lagrange equations for a non autonomous Lagrangian by constructing a k-cosymplectic structure on $\mathbb{R}^k \times T^1_k M$ defined from the Lagrangian and the canonical stable k-tangent structure on $\mathbb{R}^k \times T^1_k M$. We can also construct this k-cosymplectic structure using the *Legendre tranformation* $\mathcal{FL}$ of $\mathcal{L}$ which is the map

$$\mathcal{FL} : \mathbb{R}^k \times T^1_k M \longrightarrow \mathbb{R}^k \times (T^1_k)^* M$$

defined as follows:

If $(t, v) = (t^1, \ldots, t^k, v_1, \ldots, v_k) \in \mathbb{R}^k \times (T^1_k M)_x$ with $x \in M$ and $v_A \in T_x M$, then

$$\mathcal{FL}(t, y) = (t^1, \ldots, t^k, p^1, \ldots p^k) \in \mathbb{R}^k \times (T^1_k M)^*_x \,, \quad p^A \in T^*_x M$$

is given by

$$p^A(v_x) = (\beta_{\mathcal{L}})_A(\bar{v_x}), \quad 1 \leq A \leq k,$$

for any $v_x \in T_xM$, where $\bar{v_x} \in T_v(T^1_kM)$ is any tangent vector such that $d\tau_M(v)(\bar{v_x}) = v_x$, with $\tau_M : T^1_kM \longrightarrow M$ the canonical projection. In induced coordinates we have

$$\mathcal{FL} : (t^A, q^i, v^i_A) \longrightarrow (t^A, q^i, \frac{\partial \mathcal{L}}{\partial v^i_A}). \tag{71}$$

Now, from (64) and (71) we deduce the following.

Lemma 6.25 $(\omega_L)_A = \mathcal{FL}^*((\omega_0)_A), \quad dt^A = \mathcal{FL}^*((\eta_0)_A)$, *for all A.*

Then we have

Proposition 6.26 *The following conditions are equivalent:*

1) $\mathcal{L}$ *is regular.*

2) $\mathcal{FL}$ *is a local diffeomorphism.*

3)$(dt^A, (\omega_{\mathcal{L}})_A, V_{1,0})$ *is a k-cosymplectic structure on* $\mathbb{R}^k \times T^1_kM$.

.

7 The Cartan-Hamilton-Poincaré Form on $J^1\pi$ and $L_\pi E$

In this section we further explore relationships between n-symplectic geometry on frame bundles and multisymplectic geometry. Since $m = n + k = \dim(E)$ we will refer to the n-symplectic geometry on LE as m-symplectic geometry, and base the discussion on the n-form on $J^1\pi$ considered by Cartan, Hamilton and Poincaré. This form has various names in the literature; here we will use the name Cartan-Hamilton-Poincaré (CHP) form. Although this n-form on $J^1\pi$ has been in the literature for many years, its definition on $L_\pi E$ is relatively recent. It appeared first in [48], where the n-form was defined in terms of newly defined Cartan-Hamilton-Poincaré 1-forms. These Cartan-Hamilton-Poincaré 1-forms play the role of an m-symplectic potential on $L_\pi E$ and are discussed in Section 7.5. In Section 7.4 we give a new geometrical definition of Θ_L on $J^1\pi$. See also Section 9.4 where the Cartan-Hamilton-Poincaré 1-forms are defined using an m-symplectic Legendre transformation.

7.1 The Cartan-Hamilton-Poincaré Form on $J^1\pi$

One method used to construct the Cartan-Hamilton-Poincaré Form on $J^1\pi$ is to first construct a vector valued m-form S_ω on $J^1\pi$ associated with a volume form ω on M, as follows: For each 1-form σ on $J^1\pi$ the vector valued 1-form $S\sigma$ along $\pi_1 : J^1\pi \to M$ is defined by

$$\alpha((S\sigma)(X)) = \sigma(S_\alpha(X))$$

for any vector field X on $J^1\pi$ and any 1-form α on M. Recall S_α was defined in Section 2.4.

Now S_ω is defined according to the rule

$$S_\omega \lrcorner \ \sigma = \imath_{S\sigma}\omega$$

where $\imath_{S\sigma}$ is the derivation of type $\imath_*$ corresponding to $S\sigma$, that is

$$\sigma(S_\omega(X_1, \ldots, X_m)) = (\imath_{S\sigma}\omega)(X_1, \ldots, X_m) = \sum_{i=1}^{n} \omega((\pi_1)_* X_1, \cdots, S\sigma(X_i), \ldots, (\pi_1)_* X_m)$$

for any vector fields $X_1, \ldots, X_m$ on $J^1\pi$. In coordinates

$$S_\omega = (dy^A - y^A_j dx^j) \wedge \left(\frac{\partial}{\partial x^i} \lrcorner \ \omega \right) \otimes \frac{\partial}{\partial v^A_i} \tag{72}$$

If $\mathcal{L}_\pi : J^1\pi \to \Lambda^n M$ is a Lagrangian density, then $\mathcal{L}_\pi = \mathcal{L}\,\omega$ where $\mathcal{L} : J^1\pi \to \mathbf{R}$. The *Cartan-Hamilton-Poincaré n-form of $\mathcal{L}$* is defined by

$$\Theta_L = \mathcal{L}\,\omega + {S_\omega}^* d\mathcal{L} = \mathcal{L}\,\omega + d\mathcal{L} \circ S_\omega\,. \tag{73}$$

In coordinates

$$\Theta_{\mathcal{L}} = \mathcal{L}\,\omega + \frac{\partial \mathcal{L}}{\partial y^A_i}(dy^A - y^A_j\, dx^j) \wedge (\frac{\partial L}{\partial x^i} \lrcorner \ \omega) \tag{74}$$

7.2 The tensors S_α and S_ω on $J^1\pi$ viewed from $L_\pi E$

For each 1-form α on M, we shall define on $L_\pi E$ a tensor field $\tilde{S}_\alpha$, of type $(1,1)$ that *projects* on the tensor S_α on $J^1\pi$. Let $(B_i = B(\hat{r}_i), B_A = B(\hat{r}_A))$ be the standard vector fields of any torsion free linear connection on $\lambda : L_\pi E \to E$. In local coordinates we have

$$B_i = v^s_i \frac{\partial}{\partial x^s} + v^C_i \frac{\partial}{\partial y^C} + V_i \quad , \quad B_A = v^C_A \frac{\partial}{\partial y^C} + V_A \tag{75}$$

where V_i, V_A are vertical with respect to λ.

Now if α is an arbitrary 1-form on M and $(\pi\circ\lambda)^*\alpha$ its pull-back to $L_\pi E$, we consider on $L_\pi E$ the functions $((\pi\circ\lambda)^*\alpha)(B_i)$ for each $1\leq i\leq n$. In coordinates, if $\alpha=\alpha_r\,dx^r$, then from (75)

$$((\pi\circ\lambda)^*\alpha)(B_i)=\alpha_r\,dx^r\left(v_i^s\,\frac{\partial}{\partial x^s}+v_i^C\,\frac{\partial}{\partial y^C}+V_i\right)=\alpha_r\,v_i^r\,. \tag{76}$$

Taken together the function $\hat{\alpha}=(\alpha_a v_i^a)\hat{r}_i$ is the $(\mathbb{R}^n)^*$-valued tensorial 0-form on LE corresponding to α on M.

Definition 7.1 *The vector-valued 1-form $\tilde{S}_\alpha$ on $L_\pi E$ is defined by*

$$\tilde{S}_\alpha=((\pi\circ\lambda)^*\alpha)(B_i)\,E_B^{*i}\otimes\theta^B\quad.$$

From (36) and (76) we obtain that in local coordinates

$$\tilde{S}_\alpha=\alpha_j\,(dy^B-u_t^B\,dx^t)\otimes\frac{\partial}{\partial u_j^B}\quad. \tag{77}$$

Proposition 7.2 *The relationship between $\tilde{S}_\alpha$ on $L_\pi E$ and S_α on $J^1\pi$ is given by*

$$\tilde{S}_\alpha\,\lrcorner\,\rho_*=\rho_*\,\lrcorner\,S_\alpha$$

that is

$$\rho_*(u)(\tilde{S}_\alpha(u)(X_u))=S_\alpha(\rho(u))(\rho_*(u)(X_u))$$

for any $u\in L_\pi E$ and any $X_u\in T_u(L_\pi E)$.

Proof : It is an immediate consequence of the local expressions of $\tilde{S}_\alpha$ and S_α taking into account that $\rho^* y_t^B=u_t^B$. ■

Now, proceeding analogously, we construct a tensor field $\tilde{S}_\omega$ of type $(1,n)$ on $L_\pi E$ using the tensor field $\tilde{S}_\omega$ on $L_\pi E$, associated with a volume form ω on M. We then construct the *corresponding Cartan-Hamilton-Poincaré form* on $L_\pi E$.

For each 1-form σ on $L_\pi E$ the vector valued 1-form $\tilde{S}\sigma$ along $\pi\circ\lambda: L_\pi E\to M$ is defined by

$$\alpha((\tilde{S}\sigma)(X))=\sigma(\tilde{S}_\alpha(X)) \tag{78}$$

for any vector field X on $L_\pi E$ and any 1-form α on M. We shall compute the local expression of this 1-form. If we write

$$\sigma = \sigma_i \, dx^i + \sigma_A \, dy^A + \sigma_i^j \, du_j^i + \sigma_B^j \, du_j^B + \sigma_B^A \, du_A^B$$

and we take $\alpha = dx^j$ then from (33) and (78) we obtain

$$dx^j(\tilde{S}\sigma(\frac{\partial}{\partial x^i})) = -\sigma_B^j \, u_i^B, \quad dx^j(\tilde{S}\sigma(\frac{\partial}{\partial y^A})) = \sigma_A^j,$$
$$dx^j(\tilde{S}\sigma(\frac{\partial}{\partial u_j^i})) = dx^j(\tilde{S}\sigma(\frac{\partial}{\partial u_i^A})) = dx^j(\tilde{S}\sigma(\frac{\partial}{\partial u_B^A})) = 0$$

Therefore the local expression of $\tilde{S}\sigma$ is

$$\tilde{S}\sigma \; = \; \sigma_B^j \, (dy^B - u_t^B \, dx^t) \otimes \frac{\partial}{\partial x^j} \quad . \tag{79}$$

Definition 7.3 *The tensor field $\tilde{S}_\omega$ is defined according to the rule*

$$\tilde{S}_\omega \,\lrcorner\, \sigma = \imath_{\tilde{S}\sigma} \Omega$$

where $\imath_{S\sigma}$ is the derivation of type $\imath_*$ corresponding to $\tilde{S}\sigma$, that is

$$\sigma(\tilde{S}_\omega(X_1, \ldots, X_n)) \;=\; (\imath_{S\sigma}\omega)(X_1, \ldots, X_n) \tag{80}$$
$$= \sum_{j=1}^{n} \omega((\pi \circ \lambda)_* X_1, \ldots, \tilde{S}\sigma(X_j), \ldots, (\pi \circ \lambda)_* X_n) \tag{81}$$

for any vector fields $X_1, \ldots, X_n$ on $L_\pi E$ and any 1-form σ on $L_\pi E$.

From (79) and (80) we obtain that the local expression of $\tilde{S}_\omega$ is

$$\tilde{S}_\omega = (dy^A - u_t^A \, dx^t) \wedge \left(\frac{\partial}{\partial x^i} \,\lrcorner\, \omega \right) \otimes \frac{\partial}{\partial u_i^A} \tag{82}$$

Proposition 7.4 *The relationship between $\tilde{S}_\omega$ on $L_\pi E$ and S_ω on $J^1\pi$ is given by*

$$\tilde{S}_\omega \,\lrcorner\, \rho_* = \rho_* \,\lrcorner\, S_\omega$$

that is

$$\rho_*(u) \left(\tilde{S}_\omega(u) \left((X_u)_1, \ldots, (X_u)_n \right) \right) = S_\omega(\rho(u)) \left(\rho_*(u) \left((X_u)_1 \right), \ldots, \rho_*(u) \left((X_u)_n \right) \right)$$

for any $u \in L_\pi E$ and any $(X_u)_1, \ldots, (X_u)_n \in T_u(L_\pi E)$.

Proof : It is an immediate consequence of the local expressions (72) and (77) of $\tilde{S}_\omega$ and S_ω taking into account that $\rho^* y_t^B = u_t^B$. ■

7.3 The Cartan-Hamilton-Poincaré form Θ_L on $J^1\pi$ viewed from $L_\pi E$

Using the tensor field $\tilde{S}_\omega$ we shall construct an m-form on $L_\pi E$ that projects to the corresponding Cartan-Hamilton-Poincaré m-form on $J^1\pi$.

Definition 7.5 *A Lagrangian on $L_\pi E$ is a function $L : L_\pi E \to \mathbb{R}$.*

Definition 7.6 *[48] A Lagrangian on $L_\pi E$ is* lifted *if it satisfies the auxiliary conditions*

$$E_j^{*i}(L) = 0 \qquad E_B^{*A}(L) = 0 \tag{83}$$

Remark Using (20) these conditions imply that L is constant on the fibers of $\rho : L_\pi E \to J^1\pi$, and thus is the pull up of a function $\mathcal{L}$ on $J^1\pi$, that is $\rho^*\mathcal{L} = L$.

Definition 7.7 *If $L : L_\pi E \to \mathbb{R}$ is a lifted Lagrangian on $L_\pi E$, then we define the* Cartan-Hamilton-Poincaré m-form *of L by*

$$\theta_L = L\,\omega + \tilde{S}_\omega^* dL = L\,\omega + dL \circ \tilde{S}_\omega\,.$$

If $\omega = d^n x = dx^1 \wedge \cdots \wedge dx^n$ then from (82) we obtain that the local expression of θ_L is

$$\theta_L = \left(L - u_i^A \frac{\partial L}{\partial u_i^A} \right) d^n x + \frac{\partial L}{\partial u_i^A}\, dy^A \wedge d^{n-1} x_i\,. \tag{84}$$

Proposition 7.8 *If L is a lifted Lagrangian then the corresponding m-form satisfies $\rho^*\Theta_{\mathcal{L}} = \theta_L$, where $\Theta_{\mathcal{L}}$ is the Cartan-Hamilton-Poincaré n-form on $J^1\pi$ corresponding to $\mathcal{L}$.*

Proof It follows from the local expressions taking into account that $\rho^* y_i^A = u_i^A$. ■

7.4 The m-symplectic structure on $L_\pi E$ and the formulation of the Cartan-Hamilton-Poincaré n-form

We consider next the definition of the Cartan-Hamilton-Poincaré 1-forms on $L_\pi E$ introduced in [48, 32]. These 1-forms combine into an $\mathbb{R}^m$-valued 1-form whose exterior derivative plays the role of a general m-symplectic structure on $L_\pi E$.

Definition 7.9 *[48] Let $L : L_\pi E \to \mathbb{R}$ be a lifted Lagrangian on $L_\pi E$, and $\tau(n)$ a positive function of $n = \dim M$. The* Cartan-Hamilton-Poincaré 1-forms θ^α_L *on $L_\pi E$ are*

$$\theta^i_L = \tau(n) L \theta^i + E^{*i}_A(L)\theta^A \tag{85}$$

$$\theta^A_{\mathcal{L}} = \theta^A \tag{86}$$

where E^{*i}_A are defined above in (20), and θ^i and θ^A are the components of the canonical soldering 1-form on $L_\pi E$.

Remark The quantities $E^{*i}_A(\mathrm{L})$, referred to as the "covariant canonical momenta" in [48], are <u>*globally defined*</u> on $L_\pi E$. In local canonical coordinates (z^α, π^μ_ν), these quantities have the local expressions

$$E^{*i}_A(\mathrm{L}) = \pi^i_j p^j_B v^B_A \quad , \quad p^j_B = \frac{\partial\,\mathrm{L}}{\partial u^B_j} \tag{87}$$

and clearly are the frame components of the "canonical field momenta" $p^j_B = \frac{\partial\,\mathrm{L}}{\partial u^B_j}$. For different values of τ one can obtain the de Donder-Weyl theory [49, 44] and the Caratheodory theory [50, 44] as special cases of the formalism presented in reference [48]. The significance of these CHP 1-forms as regards other geometrical theories was also considered by MacLean and Norris. In [48] it was shown that one may construct the CHP n-form on $J^1\pi$ from the CHP 1-forms on $L_\pi E$. In this regard see also references [11, 28]. We now recall the construction of the Cartan-Hamilton-Poincaré n-form on $J^1\pi$ from these CHP 1-forms.

Proposition 7.10 *[48] Let (B_i, B_A) denote the standard horizontal vector fields of any torsion free linear connection on $\lambda : L_\pi E \to E$, and let* vol *denote the pull up to $L_\pi E$ of a fixed* *volume n-form* ω *on* M. *Set* $\mathrm{vol}_i = B_i \lrcorner\ \mathrm{vol}$. *Then when* $\tau(n) = \frac{1}{n}$ *the n-form*

$$\theta_L := \theta_L^i \wedge \mathrm{vol}_i$$

passes to the quotient to define the CHP-n-form Θ_L *on* $J^1\pi$ *associated with vol* $= \omega$.

Next we shall show here that the Cartan-Hamilton-Poincaré 1-forms can be obtained from the canonical m-tangent structure J^i, J^A on $L_\pi E$. Let $\Lambda = f_1 \wedge \cdots \wedge f_n$ be a fixed contravariant volume on M, with f_i locally written as $f_i = \alpha_i^j \frac{\partial}{\partial x^j}$. Thus Λ is a nowhere vanishing n-vector on M, which is the covariant version of a volume form on M. In coordinates

$$\lambda = det(\alpha_j^i)\, \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

Now given an arbitrary point $u = (e_i, e_A)_e$ on $L_\pi E$ we can define the n-vector

$$[\tilde{e}_i] = \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n$$

where $\tilde{e}_i = (\pi \circ \lambda)_*(u)(e_i)$. $[\tilde{e}_i]$ is a well-defined n-vector at $(\pi \circ \lambda)(u) = \pi(e) \in M$ since the vectors $\tilde{e}_i$ are linearly independent. In coordinates

$$[\tilde{e}_i] = det(v_j^i)\, \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

We can now define a function $\sigma : L_\pi E \to \mathbb{R}$ relative to the fixed contravariant volume Λ on M by the formula

$$[\tilde{e}_i] = \sigma(u)\, \lambda(\pi(e))$$

Using the local expressions above it is easy to see that in local coordinates on $L_\pi E$ one has

$$\sigma(u) = \frac{det(v_j^i)(u)}{det(\alpha_j^i(\pi(e))} \tag{88}$$

Proposition 7.11 *Let* L *be a lifted Lagrangian on* $L_\pi E$ *and let* σ *be the function defined on* $L_\pi E$ *relative to a fixed contravariant volume* Λ *on* M. *Then the Cartan-Hamilton-Poincaré* 1*-forms on* $L_\pi E$ *are given by the formula*

$$\theta_L^i = \frac{1}{\sigma}\, d_{\tilde{J}_i}(\sigma\, L)$$

where

$$\tilde{J}^i = \frac{1}{n}(E_j^{*i} \otimes \theta^j) \, + \, E_A^{*i} \otimes \theta^A$$

and $d_{\tilde{J}^i} = [\imath_{\tilde{J}^i}, d]$.

Proof From (20) and (88) we obtain that

$$E_j^{*i}(\sigma) = \sigma \, \delta_j^i \quad , \quad E_a^{*i} A(\sigma) = 0$$

and from (83) we obtain

$$E_j^{*i}(\sigma \, L) = \sigma \, L \, \delta_j^i \quad , \quad E_A^{*i}(\sigma \, L) = \sigma \, E_A^{*i}(L) \, .$$

Now from these last identities we have

$$\begin{aligned} \frac{1}{\sigma} \, d_{\tilde{J}^i}(\sigma \, L) &= \frac{1}{\sigma} \left(d(\sigma \, L) \circ \tilde{J}^i \right) = \frac{1}{\sigma} \left(\frac{1}{n} E_j^{*i}(\sigma \, L) \theta^j + E_A^{*i}(\sigma \, L) \theta^A \right) \\ &= \frac{1}{\sigma} \left(\frac{1}{n} \sigma \, L \, \delta_j^i \theta^j + \sigma \, E_A^{*i}(L) \theta^A \right) = \frac{1}{n} \, L \, \theta^i + E_A^{*i}(L) \, \theta^A \, . \quad \blacksquare \end{aligned}$$

Remark To these three constructions of the Cartan-Hamilton-Poincaré 1-forms on $L_\pi E$ we add a fourth in Section 9.4 where we show that the θ_L^α are the pull-backs, under a suitable defined m-symplectic Legendre transformation, of the canonical m-symplectic structure on LE.

8 Multisymplectic formalism

An alternative way to derive the field equations is to use the so-called multisymplectic formalism, developed by the Tulczyjew school in Warsaw (see [36, 37, 51, 52]), and independently by García and Pérez-Rendón [53, 54] and Goldschmidt and Sternberg [1]. This approach was revised by Martin [55, 56] and Gotay et al [34, 35, 57, 58, 59], and more recently by Cantrijn et al [38, 39].

8.1 Lagrangian formalism

Assume a Lagrangian $\mathcal{L} : J^1\pi \to \mathbb{R}$ where $J^1\pi$ is the 1-jet prolongation of a fibered manifold $\pi : E \to M$. M is supposed to be oriented with volume form ω. We take adapted coordinates (x^i, y^A, y_i^A) such that $\omega = dx^1 \wedge \cdots \wedge dx^n = d^n x$.

Denote $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$ where $\Theta_{\mathcal{L}}$ is the *Cartan-Hamilton-Poincaré* m-form introduced in 7.1 . From (73) we have that in local coordinates

$$\Omega_{\mathcal{L}} = d(y_i^A \frac{\partial L}{\partial y_i^A} - \mathcal{L}) \wedge d^n x - d(\frac{\partial \mathcal{L}}{\partial y_i^A}) \wedge dy^A \wedge d^{n-1}x^i$$

where $d^{n-1}x^i = \frac{\partial}{\partial x^i} \lrcorner\, \omega$.

Definition 8.1 $\Omega_{\mathcal{L}}$ *is called the* Cartan-Hamilton-Poincaré $(n+1)$*-form.*

One can use this multisymplectic form to re-express, in an intrinsic way, the *Euler-Lagrange equations*, which in coordinates take the classical form

$$\sum_{i=1}^{k} \frac{\partial}{\partial x^i}(\frac{\partial \mathcal{L}}{\partial y_i^A})(x^i, \phi^B(x), \frac{\partial \phi^B}{\partial x^i}(x)) - \frac{\partial \mathcal{L}}{\partial y^A}(x^i, \phi^B(x), \frac{\partial \phi^B}{\partial x^i}(x)) = 0, \tag{89}$$

for a (local) section ϕ of $\pi : E \to M$.

Theorem 8.2 *For a section ϕ of π the following are equivalent:*
(i) the Euler-Lagrange equations (89) hold in coordinates;
(ii) for any vector field X on $J^1\pi$

$$(j^1\phi)^*(X \lrcorner\, \Omega_{\mathcal{L}}) = 0\,. \quad \blacksquare \tag{90}$$

The proof can be found in [34].

$\Omega_{\mathcal{L}}$ is a multisymplectic form on $J^1\pi$ provided L is regular, that is, the Hessian matrix

$$(\frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial y_j^B})$$

is nonsingular.

We can extend equations (90) to sections τ of $J^1\pi \ \to M$, that is we consider sections τ such that

$$\tau^*(X \lrcorner\, \Omega_{\mathcal{L}}) = 0\,, \tag{91}$$

for any vector field X on $J^1\pi$. If the Lagrangian $\mathcal{L}$ is regular then both problems (90) and (91) are equivalent, that is, such a τ is automatically a 1-jet prolongation $\tau = j^1\phi$. Equation (91) corresponds to the so called de Donder problem (see Binz *et al* [60].)

8.2 Hamiltonian formalism

We have an exact sequence of vector bundles over E:

$$0 \to \bigwedge_1^n E \xrightarrow{i} \bigwedge_2^n E \xrightarrow{\mu} J^1\pi^* \to 0$$

where $J^1\pi^*$ is the quotient vector bundle

$$J^1\pi^* = \frac{\bigwedge_2^n E}{\bigwedge_1^n E},$$

i is the inclusion, and μ is the projection map.

$J^1\pi^*$ is sometimes defined as the affine dual bundle of $J^1\pi$ (see [17]). We have taken local coordinates (x^i, y^A, p) on $\bigwedge_1^n E$ and (x^i, y^A, p, p_A^i) on $\bigwedge_2^n E$, and then (x^i, y^A, p_A^i) can be taken as local coordinates in $J^1\pi^*$.

To develop a Hamiltonian theory, we need a Hamiltonian, in this case a section $H : J^1\pi^* \to \bigwedge_2^n E$ of the canonical projection μ. In coordinates, we have

$$H(x^i, y^A, p_A^i) = (x^i, y^A, -\hat{H}, p_A^i)$$

where $\hat{H} = \hat{H}(x^i, y^A, p_A^i) \in C^\infty(J^1\pi^*, \mathbb{R})$.

Take the pull-back $\Omega_H = H^*\Omega_E^2$ (we also have $\Theta_H = H^*\Theta_E^2$ such that $\Omega_H = -d\Theta_H$), then from (48) we have

$$\Theta_H = -\hat{H} d^n x + p_A^i dy^A \wedge d^{n-1}x^i, \quad \Omega_H = d\hat{H} \wedge d^n x - dp_A^i \wedge dy^A \wedge d^{n-1}x^i\,,$$

Ω_H is again a multisymplectic $(n+1)$-form. Now solutions of the *Hamilton equations*

$$\frac{\partial \gamma^A}{\partial x^i} = -\frac{\partial \hat{H}}{\partial p_A^i}, \qquad \sum_i \frac{\partial \gamma_A^i}{\partial x^i} = \frac{\partial \hat{H}}{\partial y^A}\,.$$

are obtained by looking for sections

$$\begin{array}{cccc} \gamma & : & M & \longrightarrow & J^1\pi^* \\ & & (x^i) & \mapsto & (x^i, \gamma^A, \gamma_A^i) \end{array}$$

such that

$$\gamma^*(Y \lrcorner\, \Omega_H) = 0$$

for any vector field Y on $J^1\pi^*$, see [38].

To relate both formalisms, we must use the Legendre transformation. For $\mathcal{L}$, we define a fibered mapping over E, $Leg : J^1\pi \longrightarrow \bigwedge_1^n E$, by

$$[Leg(j_x^1\phi)](X_1, \ldots, X_n) \;=\; (\Theta_{\mathcal{L}})_{j_x^1\phi}(\tilde{X}_1, \ldots, \tilde{X}_n)$$

for all $X_1, \ldots, X_n \in T_{\phi(x))}E$, where $\tilde{X}_1, \ldots, \tilde{X}_n \in T_{j_x^1\phi}(J^1\pi)$ are such that they project on $X_1, \ldots, X_n$, respectively.

In local coordinates

$$Leg(x^i, y^A, y_i^A) \;=\; (x^i, y^A, \mathcal{L} - y_i^A \frac{\partial \mathcal{L}}{\partial y_i^A}, \frac{\partial \mathcal{L}}{\partial y_i^A}) \,.$$

If we compose $Leg : J^1\pi \to \bigwedge_1^n E$ with $\mu : \bigwedge_1^n E \to J^1\pi^*$, we obtain the reduced Legendre transformation

$$\begin{array}{cccc} leg & : & J^1\pi & \longrightarrow & J^1\pi^* \\ & & (x^i, y^A, y_i^A) & \mapsto & (x^i, y^A, \frac{\partial \mathcal{L}}{\partial y_i^A}) \end{array}$$

which extends the usual one in mechanics, and the Legendre map defined by Günther. (see remark in Section 6.5).

A direct computation shows that $leg^*\Theta_E^2 = \Theta_{\mathcal{L}}, \quad leg^*\Omega_E^2 = \Omega_{\mathcal{L}}$.

It is clear that $leg : J^1\pi \to J^1\pi^*$ is a local diffeomorphism if and only if $\mathcal{L}$ is regular. If $\mathcal{L}$ is regular, then we can define a (local) section H as follows $H = Leg \circ leg^{-1}$

$$J^1\pi \longrightarrow \bigwedge_2^n E$$

.

Proposition 8.3 *The following assertions are equivalents:*

1) $\mathcal{L}$ is regular.

2) $\Omega_{\mathcal{L}}$ is multisymplectic, and

3) $leg : J^1\pi \to J^1\pi^*$ is a local diffeomorphism.

8.3 Ehresmann connections and the Lagrangian and Hamiltonian formalisms

A different geometric version of the field equations was given recently, based on Ehresmann connection [39].

In mechanics we look for curves and their linear approximations; that is, we look for tangent vectors. In Field Theory, we look for sections, and their linear approximations are just horizontal subspaces of Ehresmann connections in the fibration $\pi_1 : J^1\pi \to M$.

A connection in π_1 (in the sense of Ehresmann [61, 62]) is defined by a complementary distribution $\mathbf{H}$ of $V\pi_1$, i.e., we have the following Withney sum of vector bundles over E:

$$T(J^1\pi) = \mathbf{H} \oplus V\pi_1 \ .$$

As is well-known, we can characterize a connection in π_1 as a (1,1)-tensor field Γ on $J^1\pi$ such that

- $\Gamma^2 = Id$, and
- the eigenspace at the point $z \in J^1\pi$ corresponding to the eigenvalue -1 is the vertical subspace $(V\pi_1)_z$.

In other words, Γ is an almost product structure on $J^1\pi$ whose eigenvector bundle corresponding to the eigenvalue -1 is just the vertical subbundle $V\pi_1$.

We denote by

$$\mathbf{h} = \frac{1}{2}(Id + \Gamma) \ , \ \mathbf{v} = \frac{1}{2}(Id - \Gamma) \ ,$$

the horizontal and vertical projectors, respectively. Hence, the horizontal distribution is given by $\mathbf{H} = Im\, \mathbf{h}$ and $Im\, \mathbf{v} = V\pi_1$.

We say that Γ is *flat* if the horizontal distribution is integrable. In such a case, from the Frobenius theorem, there exists a horizontal local section γ of π_1 passing through each point of $J^1\pi$. Let us recall that a local section γ of $\pi_1 : J^1\pi \to M$ is called *horizontal* if it is an integral submanifold of the horizontal distribution.

Suppose that $\mathbf{h}$ is locally expressed in fibered coordinates (x^i, y^A, y_i^A) as follows:

$$\mathbf{h} = dx^i \otimes [\frac{\partial}{\partial x^i} + \Gamma_i^A \frac{\partial}{\partial y^A} + \Gamma_{ji}^A \frac{\partial}{\partial y_j^A}] \tag{92}$$

A direct computation in local coordinates shows that the equation

$$\imath_{\mathbf{h}} \Omega_{\mathcal{L}} = (n-1)\Omega_{\mathcal{L}}$$

may be considered as the geometric version of the field equations, where $\mathbf{h}$ is the horizontal projector of the Ehresmann connection in $J^1\pi \to M$. Indeed, from (92) and the local expression of $\Omega_{\mathcal{L}}$ we deduce that $\imath_{\mathbf{h}}\Omega_{\mathcal{L}} = (n-1)\Omega_L$ if and only if

$$\frac{\partial \mathcal{L}}{\partial y^A} - \frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial x^i} - \Gamma_i^B \frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial y^B} - \Gamma_{ij}^B \frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial y_j^B} + (\Gamma_i^B - y_i^B) \frac{\partial^2 \mathcal{L}}{\partial y^A \partial y_i^B} = 0\,, \tag{93}$$

$$(\Gamma_j^B - y_j^B) \frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial y_j^B} = 0\,. \tag{94}$$

If L is regular, (94) implies $\Gamma_j^B = y_j^B$, for all B, j, and then (93) becomes

$$\frac{\partial \mathcal{L}}{\partial y^A} - \frac{\partial^2 L}{\partial y_i^A \partial x^i} - y_i^B \frac{\partial^2 \mathcal{L}}{\partial y_i^A \partial y^B} - \Gamma_{ji}^B \frac{\partial^2 L}{\partial y_i^A \partial y_j^B} = 0\,, \tag{95}$$

Hence, if Γ is flat and $\gamma : M \to J^1\pi$ is a a horizontal local section locally given by $\gamma(x^i) = (x^i, \gamma^A, \gamma_i^A)$, then taking into account that $\gamma_*(T_xM) = \mathbf{H}_{\gamma(x)}$ we obtain

$$\Gamma_i^A \;=\; y_i^A \;=\; \frac{\partial \gamma^A}{\partial x^i} = \gamma_i^A\,, \qquad \Gamma_{ji}^A \;=\; \frac{\partial \gamma_j^A}{\partial x^i} \;=\; \frac{\partial^2 \gamma^A}{\partial x^i \partial x^j}\,. \tag{96}$$

This implies that γ is a 1-jet prolongation, i. e. $\gamma = j^1\phi$ and, ϕ is a solution of (95), that is, ϕ is solution of the Euler-Lagrange equations (89).

Again, we can look for Ehresmann connections in the fibration $J^1\pi^* \to M$. Indeed, if $\tilde{h}$ is the horizontal projector of such a connection, we deduce that

$$\imath_{\tilde{h}}\Omega_H = (n-1)\Omega_H$$

if and only if

$$\bigwedge\nolimits_i^A = -\frac{\partial H}{\partial p_A^i}, \qquad \sum_i \bigwedge\nolimits_{ii}^A = \frac{\partial H}{\partial y^A}\,,$$

where

$$\tilde{h} = dx^i \otimes [\,\frac{\partial}{\partial x^i} + \bigwedge\nolimits_i^A \frac{\partial}{\partial y^A} + \bigwedge\nolimits_{ji}^A \frac{\partial}{\partial p_A^j}]$$

Therefore, if $\tilde{h}$ is flat, and γ is an integral section of $\tilde{h}$, we deduce that γ satisfies the Hamilton equations for H.

8.4 Polysymplectic formalism

An alternative formalism for Classical Field Theories is the so-called polysymplectic approach (see [63, 64, 65, 66, 67, 68, 70, 71, 72, 73]). The geometric ingredients are almost the same as in multisimplectic theory, except that we consider vector-valued Cartan-Hamilton-Poincaré forms.

We start with a fibred bundle $\pi : E \to M$ as above, and introduce the following spaces

- The Legendre bundle
$$\Pi = \bigwedge^{n} M \otimes_E V^*\pi \otimes_E TM$$
where $V^*\pi$ is the dual vector bundle of the vertical bundle $V\pi$.

- The homogeneus Legendre bundle
$$Z_E = T^*E \wedge (\bigwedge^{n-1} M)\,.$$

Z_E (resp. Π) will play the role of $\bigwedge_2^m E$ (resp. $J^1\pi^*$) in multisymplectic formalism. Accordingly, we introduce coordinates (x^i, y^A, p, p_A^i) on Z_E, and (x^i, y^A, p_A^i) on Π. Moreover, there exists a canonical embedding $\theta : \Pi \to \bigwedge^{n+1} E \bigotimes_E TM$ defined by $\theta = -p_A^i\, dy^A \wedge \omega \otimes \frac{\partial}{\partial x^i}$.

Definition 8.4 *The polysymplectic form on* Π *is the unique* TM*-valued* $(n+2)$*-form* Ω *such that the relation*

$$\imath_\phi \Omega = -d(\phi \lrcorner\, \theta)$$

holds for any 1*-form* ϕ *on* M*.*

A direct computation shows that Ω has the following local expression

$$\Omega = dp_A^i \wedge dy^A \wedge \omega \otimes \frac{\partial}{\partial x^i}\,.$$

A covariant Hamiltonian is given by a Hamiltonian form, that is, a section H of the canonical projection $Z_E \to \Pi$, as in the multisymplectic settings. The field equations are provided by a connection γ in the fibration $\Pi \to M$ such that $\gamma \lrcorner\, \Omega$ is closed, and γ is then called a Hamilton connection (see [64] for details).

The Cartan-Hamilton-Poincaré m-form $\Theta_{\mathcal{L}}$ defines the Legendre transformation

$$\mathcal{FL} : J^1\pi \longrightarrow Z_E$$

by

$$\mathcal{FL}(x^i, y^A, y^A_i) = (x^i, y^A, \mathcal{L} - y^A_i \frac{\partial \mathcal{L}}{\partial y^A_i}, \frac{\partial \mathcal{L}}{\partial y^A_i}) .$$

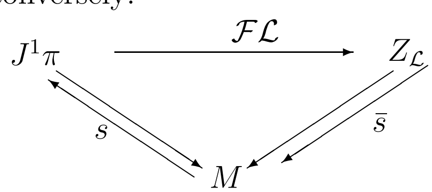
On the other hand, notice that Z_E is canonically embedded into $\bigwedge^n M$, so that it inherits the restriction Ξ_E of the canonical multisymplectic form Ω_E, say

$$\Xi_E = \Omega_{E\rfloor Z_E} .$$

Let $Z_{\mathcal{L}} = \mathcal{FL}(J^1\pi)$ and assume that it is embedded into Z_E. Therefore we have an n-form $\Xi_{\mathcal{L}}$ on $Z_{\mathcal{L}}$ which is just the restriction of Ξ_E. Of course we have

$$\Theta_{\mathcal{L}} = \mathcal{FL}^*(\Xi_{\mathcal{L}}) .$$

The Legendre morphism $\mathcal{FL}$ permits then to transport sections from the fibration $J^1\pi \to M$ to $Z_{\mathcal{L}} \to M$, and conversely:



such that, if s is a solution of the equation $s^*(X \lrcorner\ d\Theta_{\mathcal{L}}) = 0$ for all vector fields on $J^1\pi$, then $\mathcal{FL} \circ s$ is a solution of the equation $\gamma^*(\bar{X} \lrcorner\ d\Xi_{\mathcal{L}}) = 0$, for all vector fieds $\bar{X}$ on $Z_{\mathcal{L}}$, and conversely (see [64]).

In [64] is also analyzed the case of singular Lagrangians in order to compare the Hamiltonian and Lagrangian formalism.

9 n-symplectic geometry

n-symplectic geometry on frames bundles was originally developed as a generalization of Hamiltonian mechanics. The theory has, however, turned out to be a covering theory of both symplectic and multisymplectic geometries in the sense that these latter structures can be derived from n-symplectic structures on appropriate frames bundles [11, 28]. In this section we compare the n-symplectic geometry to k-symplectic/polysymplectic geometry and to multisymplectic geometry as well. Moveover we present a recent extension of the algebraic structures on an n-symplectic manifold to a general n-symplectic manifold.

9.1 The structure equations of n-symplectic geometry

The difference between n-symplectic and k-symplectic/polysymplectic geometry lies not in the properties of the canonical 2-form – they are essentially the same. Instead the real difference lies in the structure equations, the specification of LM, and the algebraic structures based on the m-symplectic Poisson bracket.

In n-symplectic geometry, one works with the soldering form on the frame bundle LM. The differential of the soldering form is a family of 2-forms that, together with the right grouping of the fundamental vertical vector fields, makes LM a m-symplectic manifold. However in n-symplectic geometry we prefer to think of $d\theta$ as a vector valued 2-form – as a single unit rather than a collection.

Recall the structure equation of m-symplectic geometry for first order observables:

$$d\hat{f}^i = -X_{\hat{f}} \lrcorner\ d\theta^i \tag{97}$$

So we have vector-valued observables ($\hat{f}^i$) and scalar-valued vector fields ($X_{\hat{f}}$), whereas the polysymplectic formalism has scalar observables and vector-valued vector fields.

In the polysymplectic formalism there exist corresponding vector fields for all functions, but these vector fields are not unique. Contrastingly, in the first order m-symplectic formalism the vector fields are unique, but only exist for a special class of functions (see Section 9.4). This uniqueness allows for the definition of Poisson brackets, which are not available in the polysymplectic formalism.

The m-symplectic formalism extends to allow higher order observables. For example, in the second order symmetric case we have:

$$d\hat{f}^{ij} = -2X_{\hat{f}}^{(i} \lrcorner\ d\theta^{j)} \tag{98}$$

Now we obtain vector-valued vector fields from an $\mathbb{R}^n \otimes_s \mathbb{R}^n$-valued function. In fact, we remark that the trace $\sum_{i=j} f^{ij}$ will satisfy the polysymplectic equation with the vector field

$-2X^i_{\hat{f}}$. In this second order case the vector fields are no longer unique, but this does not impede the definition of Poisson brackets.

For the p-th order case in m-symplectic geometry we have

$$d\hat{f}^{i_1\ldots i_p} = -p!X_{\hat{f}}{}^{(i_1\ldots i_{p-1}} \lrcorner\, d\theta^{i_p)} \qquad \text{or} \qquad d\hat{f}^{i_1\ldots i_p} = -p!X_{\hat{f}}{}^{[i_1\ldots i_{p-1}} \lrcorner\, d\theta^{i_p]} \tag{99}$$

for the symmetric and anti-symmetric cases respectively. The Poisson bracket of a p-th and a q-th order observable is a $(p+q-1)$-th order observable. The full algebra is developed in [9]. There is nothing in the polysymplectic formalism to compare to this in general.

It has been shown recently that the n-symplectic Poisson brackets defined on frame bundles extends to Poisson brackets on a general polysymplectic manifold. We present in the next sections a summary of the general results shown by Norris [32] for a general n-symplectic (polysymplectic) manifold.

9.2 General n-symplectic geometry

Let P be an N-dimensional manifold, and let $(\hat{r}_\alpha)$ denote the standard basis of $\mathbb{R}^n$, with $1 \leq n \leq N$. We suppose there exists on P a **general n-symplectic structure**, namely an $\mathbb{R}^n$-valued 2-form $\hat{\omega} = \omega^\alpha \otimes \hat{r}_\alpha$ that satisfies the following two conditions:

$$(C-1) \qquad d\omega^\alpha = 0 \quad \forall \;\; \alpha = 1, 2, \ldots, n \tag{100}$$

$$(C-2) \qquad X \lrcorner\, \hat{\omega} = 0 \quad \Leftrightarrow \quad X = 0 \tag{101}$$

Definition 9.1 *The pair $(P, \hat{\omega})$ is a general n-symplectic manifold.*

Remark In references [9, 10, 11, 12, 74, 28, 48] the term n-symplectic structure refers to the two-form that is the exterior derivative of the $\mathbb{R}^n$-valued soldering 1-form on frame bundles or subbundles of frame bundles. As outlined earlier in this paper Günther [4] was perhaps the first to consider a manifold with a non-degenerate $\mathbb{R}^n$-valued 2-form, and he used the terms *polysymplectic structure* and *polysymplectic manifold* for the non-dengenerate 2-form and manifold, respectively. In addition, when one adds two extra conditions to conditions

C-1 and C-2 one arrives at a *k-symplectic manifold*. Specifically, if P is required to support an np-dimensional distribution V such that

$$(C-3) \qquad N = p(n+1)$$

$$(C-4) \qquad \hat{\omega}|_{V\times V} = 0$$

then P is a *k-symplectic manifold* as defined by both de Leon, Salgado, et al. [5] and also by Awane [7]. To make this identification one needs to make the notational changes $n \longrightarrow k$ and $p \longrightarrow n$ in the above discussion. Thus all k-symplectic manifolds are n-symplectic, but not conversely. The example $(LE, d\hat{\theta})$ of an m-symplectic manifold introduced in Section 4.4 is also a k-symplectic manifold. On the other hand the important example of the adapted frame bundle $L_\pi E$ is m-symplectic, but not k-symplectic. The problem is that the k-symplectic dimensional requirement $N = p(m+1)$ cannot be satisifed on $L_\pi E$.

We will use the name *general m-symplectic structure* for the structure in definition 9.2 in order to emphasis the geometrical and algebraic developments that the m-symplectic approach provides. However, the definition of a general m-symplectic structure is identical with the definition of a polysymplectic structure.

9.3 Canonical coordinates

Awane [7] has proved a generalized Darboux theorem for k-symplectic geometry. Thus in the neighborhood of each point $u \in P$ one can find canonical (or Darboux) coordinates (π^α_a, z^b), $\alpha, \beta = 1, 2, \ldots k$ and $a, b = 1, 2, \ldots n$. With respect to such canonical coordinates $\hat{\omega}$ takes the form

$$\hat{\omega} = (d\pi^\alpha_a \wedge dz^a) \otimes \hat{r}_\alpha \tag{102}$$

Hence we have the following locally defined equations:

$$d\pi^\alpha_a = -\frac{\partial}{\partial z^a} \rfloor \; \omega^\alpha \,, \qquad dz^a = \frac{\partial}{\partial \pi^\alpha_a} \rfloor \; \omega^\alpha \,, \qquad (\not\Sigma_\alpha\ ) \tag{103}$$

Remark The n-symplectic approach used to characterize algebras of observables requires the existence of such canonical coordinates. From the results in [9] one knows that not all functions are *allowable n-symplectic* observables, even in the canonical case of frame bundles. Thus, for example, whether or not there exist pairs $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$, $p = 1, 2, \ldots$ that satisfy equation (105) below for a general m-symplectic manifold is an existence question. The formulas (103) will provide local examples of rank 1 solutions of the n-symplectic structure equations (105) when either the geometry is specialized to $(k = n)$-symplectic geometry where a Darboux theorem holds, or when canonical coordinates are simply known to exist. Fortunately in the case of the adapted frame bundle $L_\pi E$, canonical coordinates are known to exist.

Example: On the bundle of linear frames $\lambda : LE \to E$ one can introduce canonical coordinates in the $(z^\alpha, \pi^\alpha_\beta)$ as in ection 2.5. With respect to such a coordinate system on LE the soldering 1-form $\hat{\theta}$ has the local coordinate expression

$$\hat{\theta} = (\pi^\alpha_\beta dz^\beta) \otimes \hat{r}_\alpha \tag{104}$$

The m-symplectic 2-form $d\hat{\theta}$ clearly has the canonical form (102) in such a coordinate system.

9.4 The Symmetric Poisson Algebra Defined by $\hat{\omega}$

In this section we generalize the algebraic structures of n-symplectic geometry on frame bundles to a general n–symplectic manifold. Throughout this section we let $(P, \hat{\omega})$ be a general n-symplectic manifold as defined above. It is convenient to introduce the multi-index notation

$$\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{n-\mu}} = \hat{r}_{\alpha_1} \otimes_s \hat{r}_{\alpha_2} \otimes_s \cdots \otimes_s \hat{r}_{\alpha_{n-\mu}} \quad , \qquad 0 \le \mu \le n-1$$

In addition round brackets around indices $(\alpha\beta\gamma)$ denote symmetrization over the enclosed indices.

Definition 9.2 *For each $p \ge 1$ let SHF^p denote the set of all $(\otimes_s)^p\mathbb{R}^n$-valued functions $\hat{f} = (\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}) = (\hat{f}^{(\alpha_1\alpha_2\ldots\alpha_p)})$ on P that satisfy the equations*

$$d\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p} = -p!X_{\hat{f}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}} \lrcorner\ \omega^{\alpha_p)} \tag{105}$$

for some set of vector fields $(X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$. *We then set*

$$SHF = \oplus_{p\geq 1} SHF^p \tag{106}$$

$\hat{f} \in SHF^p$ *is a* *symmetric Hamiltonian function of rank* p.

Example: The locally defined functions $\hat{f}$ that satisfy (105) for the canonical m-symplectic manifold $(LE, d\hat{\theta})$ were given in reference [9]. In particular, contrary to the situation in symplectic geometry, not all $(\otimes_s)^p\mathbb{R}^m$-valued functions on LE are compatible with equation (105). The $p = 1, 2$ cases will clarify the structure. Let $ST^p(LE)$ denote the vector space of symmetric $(\otimes_s)^p\mathbb{R}^m$-valued GL(m)-tensorial functions on LE that correspond uniquely to symmetric rank p contravariant tensor fields on E. Similarly let $C^\infty(E, (\otimes_s)^p\mathbb{R}^m)$ denote the set of smooth $(\otimes_s)^p\mathbb{R}^m$-valued functions on LE that are constant on fibers of LE. Then

$$SHF^1 = ST^1(LE) + C^\infty(E, \mathbb{R}^m) \tag{107}$$

$$SHF^2 = ST^2(LE) + T^1(LE) \otimes_s C^\infty(E, \mathbb{R}^m) + C^\infty(E, \mathbb{R}^m \otimes_s \mathbb{R}^m) \tag{108}$$

For example, if $\hat{f} = (\hat{f}^\alpha) \in SHF^1$ and $\hat{f} = (\hat{f}^{\alpha\beta}) \in SHF^2$, then in canonical coordinates $(\pi^\alpha_\beta, z^\gamma)$ the functions $\hat{f}^\alpha$ and $\hat{f}^{\alpha\beta}$ have the general forms

$$\hat{f}^\alpha = A^a \pi^\alpha_a + B^\alpha\ , \qquad \hat{f}^{\alpha\beta} = A^{\mu\nu}\pi^\alpha_\mu \pi^\beta_\nu + B^{\mu(\alpha}\pi^{\beta)}_\mu + C^{\alpha\beta} \tag{109}$$

where A^a, B^α, $A^{\mu\nu} = A^{(\mu\nu)}$, $B^{\mu\nu}$ and $C^{\mu\nu} = C^{(\mu\nu)}$ are all constant on the fibers of $\lambda : LE \to E$ and hence are pull-ups of functions defined on E.

The analogous results for the general n-symplectic form given in (102) above are straight forward to work out in canonical coordinates. For the $p = 1$ and $p = 2$ symmetric cases, one finds:

$$\hat{f}^\alpha = \mathcal{A}^a \pi^\alpha_a + \mathcal{B}^\alpha\ , \qquad \hat{f}^{\alpha\beta} = \mathcal{A}^{ab}\pi^\alpha_a \pi^\beta_b + \mathcal{B}^{a(\alpha}\pi^{\beta)}_a + \mathcal{C}^{\alpha\beta} \tag{110}$$

where now all coefficients are functions of the coordinates z^a.

Although $\hat{\omega}$ is non-degenerate in the sense given in equation (101) above, because of the symmetrization on the right-hand-side in (105) the relationship between $\hat{f}$ and $(X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$

is not unique unless $p = 1$. Given a pair $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$ that satisfies (105) one can always add to $X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ vector fields $Y^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ that satisfy the kernel equation

$$Y^{(\alpha_1\alpha_2\ldots\alpha_{p-1}} \,\lrcorner\, \hat{\omega}^{\alpha_p)} = 0 \tag{111}$$

to obtain a new pair $(\hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}, \bar{X}_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}})$ that also satisfies (105), where

$$\bar{X}_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} = X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} + Y^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$$

Hence we associate with $\hat{f} \in SHF^p$ an equivalence class of $(\otimes_s)^{p-1}\mathbb{R}^n$-valued vector fields, which we denote by $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}]\!]$. We will see below that even though we obtain equivalence classes of Hamiltonian vector fields rather than vector fields, the geometry still carries natural algebraic structures.

Definition 9.3 *For each $p \geq 1$ let SHV^p denote the vector space of all equivalence classes of $(\otimes_s)^{p-1}\mathbb{R}^n$-valued vector fields $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}]\!]$ on P that satisfy the equations (105) for some $\hat{f} = \hat{f}^{\alpha_1\alpha_2\ldots\alpha_p}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_p} \in SHF^p$. We then set*

$$SHV = \oplus_{p\geq 1} SHV^p \tag{112}$$

$[\![\hat{X}_{\hat{f}}]\!]$ will be referred to as the generalized rank p Hamiltonian vector field defined by $\hat{f}$.

Example: The Hamiltonian vector field $X_{\hat{f}}$ for the rank 1 element in (109) is unique, and has the form

$$X_{\hat{f}} = A^\alpha \frac{\partial}{\partial z^\alpha} - \left(\frac{\partial A^\beta}{\partial z^\gamma}\pi^\alpha_\beta + \frac{\partial B^\alpha}{\partial z^\gamma}\right)\frac{\partial}{\partial \pi^\alpha_\gamma} \tag{113}$$

The equivalence class of $\mathbb{R}^m$-valued Hamiltonian vector fields corresponding to the rank 2 element in (109) on LE has representatives of the form

$$X_{\hat{f}}{}^\alpha = (A^{\mu\nu}\pi^\alpha_\mu + B^{\nu\alpha})\frac{\partial}{\partial z^\nu} - \frac{1}{2}\left(\frac{\partial A^{\mu\beta}}{\partial z^\gamma}\pi^\alpha_\mu\pi^\nu_\beta + \frac{\partial B^{\mu(\alpha}}{\partial z^\gamma}\pi^{\nu)}_\mu + \frac{\partial C^{\alpha\nu}}{\partial z^\gamma}\right)\frac{\partial}{\partial \pi^\nu_\gamma} + Y^{\alpha\nu}_\gamma \frac{\partial}{\partial \pi^\nu_\gamma} \tag{114}$$

where $Y^{\alpha\beta}_\gamma$ are functions subject to the constraint

$$Y^{(\alpha\beta)}_\gamma = 0$$

but are otherwise completely arbitrary. The fact that $Y^\alpha = Y^{\alpha\mu}_\nu \frac{\partial}{\partial \pi^\mu_\nu}$ is purely vertical on $\lambda : LE \to E$ follows from (111).

For the n-symplectic rank 2 symmetric observable given above in (110), one can check easily that the local coordinate form of a representative $X^{\alpha}_{\hat{f}}$ of the equivalence class of Hamiltonian vector fields $[\![\hat{X}_{\hat{f}}]\!]^{\alpha}$ that satisfies (105) has the form

$$X^{\alpha} = (\mathcal{A}^{ab}\pi^{\alpha}_{a} + \mathcal{B}^{b\alpha})\frac{\partial}{\partial z^{b}} - \frac{1}{2}\left(\frac{\partial \mathcal{A}^{ab}}{\partial z^{d}}\pi^{\alpha}_{a}\pi^{\sigma}_{b} + \frac{\partial \mathcal{B}^{a(\alpha}}{\partial z^{d}}\pi^{\sigma)}_{a} + \frac{\partial \mathcal{C}^{\alpha\sigma}}{\partial z^{d}}\right)\frac{\partial}{\partial \pi^{\sigma}_{d}} + Y^{\alpha} \tag{115}$$

9.4.1 Poisson Brackets

We show that the n-symmetric Poisson brackets defined on frame bundles can also be defined in a general n-symplectic manifold.

Definition 9.4 *For $p, q \geq 1$ define a map $\{\ ,\ \} : SHF^{p} \times SHF^{q} \to SHF^{p+q-1}$ as follows. For $\hat{f} = f^{\alpha_1\alpha_2\ldots\alpha_p}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_p} \in SHF^{p}$ and $\hat{g} = g^{\beta_1\beta_2\ldots\beta_q}\hat{r}_{\beta_1\beta_2\ldots\beta_q} \in SHF^{q}$*

$$\{\hat{f}, \hat{g}\}^{\alpha_1\alpha_2\ldots\alpha_{p+q-1}} := p! X_{\hat{f}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}\ldots\alpha_{p+q-1})}\right) \tag{116}$$

where ${X_{\hat{f}}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ is any set of representatives of the equivalence class $[\![\hat{X}_{\hat{f}}]\!]$.

We need to make certain that $\{\hat{f}, \hat{g}\}$ is well-defined. Suppose we have two representatives $X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ and $\bar{X}_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} = X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}} + Y^{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ of $[\![\hat{X}_{\hat{f}}]\!]$. Then it follows easily from (111) that

$$\bar{X}_{\hat{f}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}\ldots\alpha_{p+q-1})}\right) = {X_{\hat{f}}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}}\left(\hat{g}^{\alpha_p\alpha_{p+1}\ldots\alpha_{p+q-1})}\right)$$

Hence the bracket is independent of choice of representatives. That $\{\hat{f}, \hat{g}\}$ actually is in SHF^{p+q-1} will follow from Corollary (9.7) below.

Definition 9.5 *Let $[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}]\!]$ and $[\![\hat{X}_{\hat{g}}]\!] = [\![X_{\hat{g}}^{\alpha_1\alpha_2\ldots\alpha_{p-1}}\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}]\!]$ denote the equivalence classes of vector-valued vector fields determined by $\hat{f} \in SHF^{p}$ and $\hat{g} \in SHF^{q}$, respectively. Define a bracket $[\![\ ,\ ]\!] : SHV^{p} \times SHV^{q} \to SHV^{p+q-1}$ by*

$$[\![[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]\!] \quad = \quad [\![[{X_{\hat{f}}}^{(\alpha_1\alpha_2\ldots\alpha_{p-1}}\ ,\ {X_{\hat{g}}}^{\alpha_p\alpha_{p+1}\ldots\alpha_{p+q-2})}]\hat{r}_{\alpha_1\alpha_2\ldots\alpha_{p+q-2}}]\!] \tag{117}$$

where the "inside" bracket on the right-hand side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives. (Notice the symmetrization over all the upper indices in this equation.)

We again need to show that this bracket is well-defined. This is shown in the following lemma, in which we will need the formula

$$L_{X^{(J}}\omega^{\alpha)} = 0 \tag{118}$$

which follows easily from (105) and the formula $L_X\omega = X \lrcorner\ d\omega + d(X \lrcorner\ \omega)$. In (118) J denotes the multiindex $\alpha_1\alpha_2\ldots\alpha_{p-1}$, and X^J denotes a representative of a rank p Hamiltonian vector field satisfying equations (105). The next lemma shows that the bracket defined in (117) is (i) independent of choice of representatives, and (ii) closes on the set of equivalence classes of vector-valued Hamiltonian vector fields. The proof of the lemma can be found in [32], which is quite similar to the proof of the analogous result in symplectic geometry.

Lemma 9.6 *Let $[\![\hat{X}_{\hat{f}}]\!]$ and $[\![\hat{X}_{\hat{g}}]\!]$ denote the equivalence classes of vector-valued vector fields determined by $\hat{f} \in SHF^p$ and $\hat{g} \in SHF^q$, respectively. Then*

$$[\![[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]]\!] = \frac{(p+q-1)!}{p!\ q!}[\![\hat{X}_{\widehat{\{\hat{f},\hat{g}\}}}]\!] \tag{119}$$

Corollary 9.7

$$\{\hat{f}, \hat{g}\} \in SHF^{p+q-1}$$

Theorem 9.8 $(SHV, [\![\ ,\ ]\!])$ *is a Lie Algebra.*

Proof The bracket defined in (117) is clearly anti-symmetric. To check the Jacobi identity we note that we only need check it for arbitrary representatives, and we may use the very definition (117) for the calculation. Since the "inside" bracket on the right-hand-side in (117) is the ordinary Lie bracket for vector fields, we see that the bracket defined in (117) also must obey the identiy of Jacobi. ■

We can now show that SHF is a Poisson algebra under the bracket defined in (116).

Theorem 9.9 $(SHF, \{\ ,\ \})$ *is a Poisson algebra over the commutative algebra* $(SHF, \otimes_s)$.

Proof The symmetrized tensor product $\otimes_s$ makes SHF into a commutative algebra. If we now consider again elements $\hat{f} \in SHF^p$, $\hat{g} \in SHF^q$ and $\hat{h} \in SHF^r$, then by using definition (116) one may show that

$$\{\hat{f}, \hat{g} \otimes_s \hat{h}\} = \{\hat{f}, \hat{g}\} \otimes_s \hat{h} + \hat{g} \otimes_s \{\hat{f}, \hat{h}\} \quad . \tag{120}$$

Thus the bracket defined in (116) acts as a derivation on the commutative algebra. ■

Example: In the canonical case $P = LE$ the Poisson brackets just defined have a well-known interpretation. As mentioned above the homogeneous elements in SHF^p make up the space $ST^p(LE)$, the symmetric rank p GL(m)-tensorial functions that correspond to symmetric rank p contravariant tensor fields on E. Then $ST = \oplus_{p\geq 1} ST^p \subset SHF$, and the bracket $\{\ ,\ \} : ST^p \times ST^q \to ST^{p+q-1}$ has been shown [12] to be the frame bundle version of the Schouten-Nijenhuis bracket of the corresponding symmetric tensor fields on E.

There is also a Schouten-Nijenhuis bracket for anti-symmetric contravariant tensor fields on E, and as one might expect this bracket also extends to LE. This leads to a graded m-symplectic Poisson algebra of anti-symmetric tensor-valued functions on LE [11].

9.5 The Legendre Transformation in m-symplectic theory on $L_\pi E$

One can define the CHP 1-forms, defined above in Definition 7.9, using a frame bundle version of the Legendre transformation. Given a lifted Lagrangian $\mathrm{L} : L_\pi E \to \mathbb{R}$ we obtain a mapping $\phi_{\mathrm{L}} : L_\pi E \to LE$ given by

$$\phi_{\mathrm{L}}(u) = \phi_{\mathrm{L}}(e, e_i, e_A) = \left(e, \frac{1}{\tau\,\mathrm{L}(u)} e_i, e_A - \frac{1}{\tau\,\mathrm{L}(u)} E^{*a}_A(\mathrm{L})(u) e_a\right) \tag{121}$$

The condition that this mapping end up in LE is that the Lagrangian be **non-zero**, and *for the rest of this paper we will assume this condition.* We refer to this mapping as the *m-symplectic Legendre transformation.* Our goal is to prove Theorem (9.13), namely that $\hat{\theta}_{\mathrm{L}} = \phi^*_{\mathrm{L}}(\hat{\theta})$ where $\hat{\theta}$ is the canonical soldering 1-form on the image Q_{L} of ϕ_{L}.

To clarify the meaning of the Legendre transformation (121) we introduce a new manifold $\tilde{P}$ as follows. Let J denote the subgroup of $GL(n)$ consisting of matrices of the form

$$\begin{pmatrix} I & \xi \\ 0 & I \end{pmatrix} \qquad \xi \in \mathbb{R}^{n\times k}$$

Define $\tilde{P}$ by

$$\tilde{P} = L_\pi E \cdot J = \{(e_i, e_A + \xi^j_A e_j) \mid (e_i, e_A) \in L_\pi E \ , \ \xi \in \mathbb{R}^{n\times k}\} \tag{122}$$

We collect together the pertinent results that are proved in [48, 32] and that lead up to Theorem (9.13)

Lemma 9.10 *$\tilde{P}$ is a open dense submanifold of the bundle of frames LE of E.*

Lemma 9.11 *There is a canonical diffeomorphism from $\tilde{P}$ to the product manifold $L_\pi E \times \mathbb{R}^{m\times k}$.*

Using this fact one can the prove the following lemma. We let Q_L denote the range of the Legendre transformation.

Lemma 9.12 *If the Lagrangian L is non-zero, then the Legendre transformation $\phi_{\mathrm{L}} : L_\pi E \to Q_{\mathrm{L}}$ is a diffeomorphism.*

These facts taken together lead to the following fundamental theorem:

Theorem 9.13 *Let L be the pull-up to $L_\pi E$ of a non-zero Lagrangian on $J^1\pi$, and let ϕ_{L} denote the m-symplectic Legendre transformation defined above in (121). Then*

$$\hat{\theta}_{\mathrm{L}} = \phi^*_{\mathrm{L}}(\hat{\theta}) \tag{123}$$

Proof The proof is a direct calculation using the definition (121).

Remark This theorem has an obvious analogue in symplectic mechanics, where the symplectic form on the velocity phase space TE is, for a regular Lagrangian, the pull back under the Legendre transformation of the canonical 1-form on T^*M. There is also a similar theorem in *multisymplectic geometry* where the CHP m-form on $J^1\pi$ is known [34] to be the pull back of the canonical multisymplectic m-form on $J^{1*}\pi$.

Now Q_{L}, being a submanifold of LE, supports the restriction $\hat{\theta}|_{Q_{\mathrm{L}}}$ of the $\mathbb{R}^m$-valued soldering 1-form $\hat{\theta}$. It is easy to verify that the closed $\mathbb{R}^m$-valued 2-form $d\hat{\theta}|_{Q_{\mathrm{L}}}$ is also non-degenerate, and hence $(Q_{\mathrm{L}}, d(\hat{\theta}|_{Q_{\mathrm{L}}}))$ is an m-symplectic manifold. Using the fact that Q_{L} and

$L_\pi E$ are diffeomorphic under the Legendre transformation, we obtain the following corollary to Theorem 9.13.

Corollary 9.14 $(L_\pi E, d\hat{\theta}_{\mathrm{L}})$ *is an m-symplectic manifold.*

To find the *allowable observables* of this theory one can set up [32] the equations of m-symplectic reduction to find the subset of m-symplectic observables on LE that reduce to the submanifold Q_{L}.

9.6 The Hamilton-Jacobi and Euler-Lagrange equations in m-symplectic theory on $L_\pi E$

Working out the local coordinate form of the CHP-1-forms, given in Definition 7.9, in Lagrangian coordinates one finds

$$\theta^i_L = -H^i_j dx^j + P^i_A dy^A \quad (124)$$

$$\theta^A_L = P^A_j dx^j + P^A_B dy^B \quad (125)$$

where

$$H^i_j = u^i_k(p^k_B u^B_j - \tau(n) L \delta^k_j) \quad (126)$$

$$P^i_B = u^i_k p^k_B \quad (127)$$

$$P^A_j = -u^A_B u^B_j \quad (128)$$

$$P^A_B = u^A_B \quad (129)$$

The H^i_j are the components of the **covariant Hamiltonian**, and the P^i_B are the components of the **covariant canonical momentum** [48]. Defining symbols h^k_j by the formula

$$h^k_j = p^k_B u^B_j - \tau(n) L \delta^k_j \quad (130)$$

the covariant Hamiltonian (126) can be expressed as $H^i_j = u^i_k h^k_j$. Setting $\tau(n) = 1$ one finds that h^i_j has the form of Carathéodory's Hamiltonian tensor [44, 50]. Similarly, setting $\tau = \frac{1}{n}$ one finds that $h = h^i_i$ yields the Hamiltonian in the de Donder-Weyl theory [44, 49].

9.6.1 The m-symplectic Hamilton-Jacobi Equation on $L_\pi E$

The Carathéodory-Rund and de Donder-Weyl Hamilton-Jacobi equations occur as special cases of a general Hamilton-Jacobi equation that can be set up on $L_\pi E$. Proceeding by analogy with the time independent Hamilton-Jacobi theory we seek Lagrangian submanifolds of $L_\pi E$. However, since the dimension of $L_\pi E$ is in general not twice the dimension of E, a new definition is needed. For our purposes here we will consider $m = n + k$ dimensional submanifolds of $L_\pi E$ that arise as sections of λ. In particular we consider sections $\sigma : E \rightarrow L_\pi E$ that satisfy

$$\sigma^*(d\theta^\alpha_L) = 0 \tag{131}$$

These are the *m-symplectic Hamilton-Jacobi equations* [48].

Since $\sigma^*(d\theta^\alpha_L) = d\left(\sigma^*(\theta^\alpha_L)\right)$ the condition (131) asserts that the 1-forms $\sigma^*(\theta^\alpha_L)$ are locally exact, and we express this as

$$\sigma^*(\theta^\alpha_L) = dS^\alpha \tag{132}$$

in terms of $m = n + k$ new functions S^α defined on open subsets of E. For convenience we will denote objects on $L_\pi E$ pulled back to E using σ with an over-tilde. Thus, for example, $\tilde{H}^i_j = H^i_j \circ \sigma$ and $\tilde{P}^i_A = P^i_A \circ \sigma$. Then we get from (126)–(129) and (132)

$$\textbf{(a)}\;\; \tilde{H}^i_j \;=\; -\frac{\partial S^i}{\partial x^j}\,, \qquad \textbf{(b)}\;\; \tilde{P}^i_A = \frac{\partial S^i}{\partial y^A} \tag{133}$$

$$\textbf{(a)}\;\; \tilde{u}^A_B \tilde{u}^B_j \;=\; -\frac{\partial S^A}{\partial x^j}\,, \qquad \textbf{(b)}\;\; \tilde{u}^A_B = \frac{\partial S^A}{\partial y^B} \tag{134}$$

Recalling that $H^i_j = P^i_B u^B_j - \tau(n) L u^i_j$ and P^i_A are functions of the coordinates x^i, y^A, u^i_j and u^A_i, equations (133) can be combined into the single equation

$$H^i_j(x^a, y^B, u^a_b, u^B_a, \frac{\partial S^i}{\partial y^B}) \circ \sigma = -\frac{\partial S^i}{\partial x^j} \tag{135}$$

Similarly combining equations (134) we obtain

$$\frac{dS^A}{dx^j} = 0$$

We next consider special cases of these m-**symplectic Hamilton-Jacobi equations**.

9.6.2 The Theory of Carathéodory and Rund

We note from (126), (127), and (130) that $H^i_j = u^i_k h^k_j$ and $P^i_A = u^i_k p^k_A$, where the matrix of functions (u^i_j) is GL(n)-valued. Using the notation $P^i_j = -H^i_j$ and $\tilde{u}^i_j = u^i_j \circ \sigma$ we may rewrite (126) and (127) in the form

$$\tilde{P}^i_j = -\tilde{u}^i_k \tilde{h}^k_j \ , \qquad \tilde{P}^i_A = \tilde{u}^i_k \tilde{p}^k_A \tag{136}$$

If we take $t(n) = 1$ then these equations are the equations defining the *canonical momenta* in Rund's canonical formalism for Carathéodory's geodesic field theory (see equations (1.22), page 389 in [44], with the obvious change in notation). In this situation equation (135) can be identified with the Rund's Hamilton-Jacobi equation for Carathéodory's theory (see equation (3.29) on page 240 in [44]). We recall [44] that one can derive the Euler-Lagrange field equations from this Hamilton-Jacobi equation.

In (136) we have the result that the arbitrary non-singular matrix-valued functions $(\tilde{u}^i_j)$ that occur in Rund's canonical formalism for Carathéodory's theory have a geometrical interpretation in the present setting. Specifically they correspond to the coordinates for linear frames for M. These defining relations are derived from Rund's **transversality condition**, and we now show that this condition has the elegant reformulation as the kernel of $(\theta^i_{\mathcal{L}})$.

We will say that a vector X at $e \in E$ is transverse to a solution surface through e that is defined by a given Lagrangian L, if $X = d\lambda(\hat{X})$, where $\hat{X} \in T_u(L_\pi E)$ satisfies $\hat{X} \lrcorner \ \theta^i_L = 0$, for some $u \in \lambda^{-1}(e)$. $\hat{X}$ thus satisfies the equations

$$0 = -H^i_j X^j + P^i_A X^A = u^i_k \left(-h^k_j X^j + p^k_A X^A\right)$$

$$X^j = \hat{X}(x^j) \ , \quad X^A = \hat{X}(y^A)$$

from which we infer

$$0 = -h^k_j X^j + p^k_A X^A \tag{137}$$

This is Rund's transversality condition for the theory of Carathéodory when we take $\tau(n) = 1$ (see equation (1.10), page 388 in [44]). The canonical momenta P^i_j and P^i_A are defined by Rund to be solutions of

$$0 = P^i_j X^j + P^i_A X^A \tag{138}$$

when (X^j, X^A) satisfy (137). Rund's solutions of these equations are given in (136). Looking at (136), (137) and (138) we see that the introduction of the u^i_j in (136) amounts to the introduction of the GL(n) freedom for linear frames for M.

9.6.3 de Donder-Weyl Theory

Returning to (135) let us reduce this equation by making several assumptions. We suppose that $\mathcal{L}$ is regular (in the usual sense on $J^1\pi$), that the section σ is such that $\tilde{u}^i_j = \delta^i_j$, and we make the choice $\tau(n) = \frac{1}{n}$. Now summing $i = j$ in (135) we obtain

$$\tilde{h}(x^i, y^B, \frac{\partial S^i}{\partial y^B}) = -\frac{\partial S^i}{\partial x^i}$$

where $\tilde{h} = \tilde{p}^i_A \tilde{u}^A_i - \tilde{L}$. This equation is the Hamilton-Jacobi equation of the de Donder-Weyl theory, as presented by Rund (see equation (2.31) on page 224 in [44]). We recall [44] that one can derive in this case also the Euler-Lagrange field equations from the de Donder-Weyl Hamilton-Jacobi equation.

9.7 Hamilton Equations in m-symplectic geometry

The structure of equations (124) - (127) suggests that one should be able to derive generalized Hamilton equations if the canonical momenta $p^i_A = \frac{\partial L}{\partial u^A_i}$ can be introduced as part of a local coordinate system on $L_\pi E$. Part of the original philosophy used in developing m-symplectic geometry in reference [9] was to switch from scalar equations to tensor equations, motivated by the fact that the soldering 1-form is vector-valued. In particular, the basic structure equation (97) in m-symplectic geometry is tensor-valued. We show next that

$$u^*(\eta \lrcorner\, d\theta^i_L) = 0 \tag{139}$$

where $u : M \to L_\pi E$ is a section of $\pi \circ \lambda$, and η is any vector field on $L_\pi E$, yields generalized canonical equations that contain known canonical equations as special cases. We consider here only $d\theta^i_L$ since by Proposition (7.10) it alone is needed to construct the CHP-m-form on $J^1\pi$.

We need the following definition in order to introduce the canonical momenta as part of a coordinate system on $L_\pi E$.

Definition 9.15 *A Lagrangian L on $L_\pi E$ is* **regular** *if the $(n+k)\times(n+k)$ matrix*

$$\left(E^{*i}_A \circ E^{*j}_B(L)\right)$$

is non-singular.

Working out the terms of this matrix in Lagrangian coordinates using (20) we obtain

$$E^{*i}_A \circ E^{*j}_B(L) = u^j_a u^i_b v^E_B v^D_A \left(\frac{\partial^2 L}{\partial u^E_a \partial u^D_b}\right)$$

It is clear that this definition is equivalent to the standard definition of regularity on $J^1\pi$.

We now consider the transformation of coordinates from the set $(x^i, y^A, u^i_j, u^A_k, u^A_B)$ to the new set $(\bar{x}^i, \bar{y}^A, \bar{u}^i_j, p^j_A, \bar{u}^A_B)$ where

$$\bar{x}^i = x^i \ , \quad \bar{y}^A = y^A \ , \quad \bar{u}^i_j = u^i_j \ , \quad \bar{u}^A_B = u^A_B \ , \quad p^i_A = \frac{\partial L}{\partial u^A_i}$$

Computing the Jacobian one finds that the new barred functions will be a proper coordinate system whenever the Lagrangian is regular. For the remainder of this section we shall assume that L has this property, despite the fact that many important examples (see [34, 35]) have non-regular Lagrangians. Moreover, for simplicity we will drop the bars on the new coordinates.

In the generalized canonical equation (139) we now take $\eta = \frac{\partial}{\partial p^i_A}$. We find the result

$$0 = \left(\frac{\partial H^j_k}{\partial p^i_A} \circ u\right) + (u^j_i \circ u)\left(\frac{\partial (y^A \circ u)}{\partial x^k}\right)$$

Using $H^j_k = u^j_i h^i_k$ and the fact that (u^i_j) is a non-singular matrix valued function, this last equation reduces to

$$\frac{\partial h^j_k}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^k} \delta^j_i$$

This is our first set of m**-symplectic Hamilton equations**. Notice that by summing $j = k$ in this equation we obtain

$$\frac{\partial h}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^i} \tag{140}$$

Upon setting $\tau(n) = \frac{1}{n}$ we obtain half of the de Donder-Weyl canonical equations. Under suitable but complicated conditions these equations, with $\tau(n) = 1$, will also reproduce part of Rund's canonical equations for the theory of Carathéodory.

In the generalized canonical equation (139) we now take $\eta = \frac{\partial}{\partial y^A}$. We find

$$0 = u^* \left(d(u^i_k p^k_A) + u^i_k \frac{\partial h^k_j}{\partial y^A} dx^j \right)$$

Using an "over bar" notation to denote objects pulled back to M by u we may write this as

$$\frac{\partial}{\partial x^j} \left(\bar{u}^i_k \bar{p}^k_A \right) = -\bar{u}^i_k \left(\frac{\partial h^k_j}{\partial y^A} \right) \circ u \tag{141}$$

This is our second set of m-**symplectic Hamilton equations**.

Notice that what is non-standard in (141) is the appearance of the derivatives of the functions $\bar{u}^i_j = u^i_j \circ u$. If, however, the section $u : M \to L_\pi E$ is such that the $\bar{u}^i_j$ are constants, then these equations reduce to

$$\frac{\partial(\bar{p}^k_A)}{\partial x^j} = -\frac{\partial h^k_j}{\partial y^A} \circ u$$

Setting $\tau(n) = \frac{1}{n}$ and summing $k = j$ in this equation we obtain

$$\frac{\partial(\bar{p}^i_A)}{\partial x^i} = -\frac{\partial h}{\partial y^A} \circ u$$

These equations, together with equations (140) when $\tau(n) = \frac{1}{n}$, are the complete canonical equations in the de Donder-Weyl theory.

Acknowledgments

This work was Partially supported by grants DGICYT (Spain) PB97-1257, PGC2000-2191-E and PGIDT01PXI20704PR.

References

[1] H. Goldschmidt, S. Sternberg: The Hamilton-Cartan formalism in the calculus of variations. *Ann. Inst. Fourier* **23** (1973), 203-267.

[2] M. Bruckheimer: Ph.D. dissertation, University of Southampton, 1960.

[3] R.S. Clark, D.S. Goel: An almost cotangent manifolds. *J. Differential Geom.* **9** (1974), 109-122.

[4] Ch. Günther: The polysymplectic Hamiltonian formalism in field theory and calculus of variations I: The local case. *J. Differential Geom.* **25** (1987), 23-53.

[5] M. de León, I. Méndez, M. Salgado: p-Almost cotangent structures. *Boll. Unione Mat. Ital.* (7) **7-a** (1993), 97-107.

[6] M. de León, I. Méndez, M. Salgado: Regular p-almost cotangent structures. *J. Korean Math. Soc.* **25**, (1988), No.2, 273-287.

[7] A. Awane: k-symplectic structures, *J. Math. Phys.* **33** (1992), 4046-4052.

[8] A. Awane: G-spaces k-symplectic homogènes, *J. Geom. Phys.* **13** (1994), 139-157.

[9] L.K. Norris: Generalized symplectic geometry on the frame bundle of a manifold, Lecture given at the AMS Summer Research Institute on Differential Geometry, 1990, at U.C.L.A.

[10] L.K. Norris: Generalized symplectic geometry on the frame bundle of a manifold, *Proc. Symp. Pure Math.* **54**, Part 2 (Amer. Math. Soc., Providence RI, 1993), 435-465.

[11] L.K. Norris: Symplectic geometry on T^*M derived from n-symplectic geometry on LM. *J. Geom. Phys.* **13** (1994), 51-78.

[12] L.K. Norris: Schouten-Nijenhuis Brackets. *J. Math. Phys.* **38** (1997), 2694-2709.

[13] R.S. Clark, M. Bruckheimer: Sur les estructures presque tangents. *C. R. Acad. Sci. Paris Sér . I Math.* **251** (1960), 627-629.

[14] H.A. Eliopoulos: Structures presque tangents sur les variétés différentiables. *C. R. Acad. Sci. Paris Sér . I Math.* **255** (1962), 1563-1565.

[15] M. de León, I. Méndez, M. Salgado: p-almost tangent structures. *Rend. Circ. Mat. Palermo* Serie II **XXXVII** (1988), 282-294.

[16] M. de León, I. Méndez, M. Salgado: Integrable p–almost tangent structures and tangent bundles of p^1-velocities. *Acta Math. Hungar.*, Vol. 58 (1-2) (1991), 45-54.

[17] D.J. Saunders: *The Geometry of Jet Bundles*. Cambridge University Press, Cambridge, 1989.

[18] F. Brickell, R.S. Clark: Integrable almost tangent structures. *J. Differential. Geom.* **9** (1974), 557-563.

[19] R. S. Clark, D.S. Goel: On the geometry of an almost tangent structure. *Tensor (N. S.)* **24** (1972), 243-252.

[20] M. Crampin, G. Thompson: Affine bundles and integrable almost tangent structures. *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 61-67.

[21] M. Crampin: Tangent bundle geometry for Lagrangian dynamics. *J. Phys. A: Math. Gen.* **16** (1983), 3755–3772.

[22] M. Crampin: Defining Euler-Lagrange fields in terms of almost tangent structures. *Phys. Lett. A* **95** (1983), 466-468.

[23] J. Grifone: Structure presque-tangente et connexions, I. *Ann. Inst. Fourier* **22** (1972), 287-334.

[24] J. Grifone: Structure presque-tangente et connexions, II. *Ann. Inst. Fourier* **22** (1972), 291-338.

[25] J. Klein: Espaces variationelles et mécanique. *Ann. Inst. Fourier* **12** (1962),1-124.

[26] G. Thompson, U. Schwardmann: Almost tangent and cotangent structures in the large. *Trans. Amer. Math. Soc.* **327** (1991), 313-328.

[27] A. Morimoto: Liftings of some types of tensor fields and connections to tangent p^r-velocities. *Nagoya Math. J.* **40** (1970), 13-31.

[28] R.O. Fulp, J.K. Lawson, L.K. Norris: Generalized symplectic geometry as a covering theory for the Hamiltonian theories of classical particles and fields. *J. Geom. Phys.* **20** (1996), 195-206.

[29] J.K. Lawson: *Generalized symplectic geometry for classical fields and spinors*, Ph.D. dissertation, Dept. Math., North Carolina State Univ., Raleigh, 1994.

[30] S. Kobayashi, K. Nomizu: *Foundations of differential geometry*, Vol. I. Interscience, New York, 1963.

[31] G. Thompson: Integrable almost cotangent structures and Legendrian bundles. *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 61-67.

[32] L.K. Norris: The n-symplectic Algebra og Observables in Covariant Lagrangian field Theory. To appear in *J. Math. Phys.*

[33] M. de León, E. Merino, M. Salgado: k-cosymplectic manifolds and Lagrangian field theories. *J. Math. Phys.* **42** (2001), .

[34] M. Gotay, J. Isenberg, J. Marsden: *Momentum Maps and Classical Relativistic Fields, Part I: Covariant Field Theory*, 1997, MSRI Preprint.

[35] M. Gotay, J. Isenberg, J. Marsden: *Momentum Maps and Classical Relativistic Fields, Part II: Canonical Analisys of Field Theories*, 1999, MSRI Preprint.

[36] J. Kijowski, W. Szczyrba: Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory. Géométrie symplectique et physique mathématique (Colloq. Internat. C.N.R.S., Aix-en-Provence, 1974), pp. 347-349.

[37] J. Kijowski, W. M. Tulczyjew: *A symplectic framework for field theories*. Lecture Notes in Physics, 107. Springer-Verlag, New York, 1979.

[38] F. Cantrijn, A. Ibort, M. de León: On the geometry of multisymplectic manifolds. *J. Austral. Math. Soc. Ser. A* **66** (1999), 303-330.

[39] F. Cantrijn, A. Ibort, M. de León: Hamiltonian structures on multisymplectic manifolds. *Rend. Sem. Mat. Univ. Politec. Torino*, **54** (1996), 225-236.

[40] J.F. Carinena, M. Crampin, A. Ibort: On the multisymplectic formalism for first order field theories. *Diff. Geom. Appl.* **1** (1991), 345-374.

[41] M. de León, E. Merino, J Oubina, P. Rodrigues, M. Salgado: Hamiltonian systems on k-cosymplectic manifolds, *J. Math. Phys,*, **39** , 91997), 876-893.

[42] A. Echevarría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Multivector fields and connections: Setting Lagrangian equations in field theories. *J. Math. Phys.* **39** (1998), 4578-4603.

[43] A. Echevarría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Multivector field formulation of Hamiltonian field theories: equations and symmetries. *J. Phys. A: Math. Gen.* **32** (1999), 8461-8484.

[44] H. Rund: *The Hamilton-Jacobi Theory in the Calculus of Variations*, Reprinted Edition, Robert E. Krieger Publishing Co. Inc., Huntington, New York, 1973.

[45] F. Cantrijn, A. Ibort, M. de León: Gradient vector fields on cosymplectic manifolds, *J. Phys. A: Math. gen.* **25** (1992), 175-188,

[46] A. Albert: Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact. *J. Geom. Phys.* **6** (1989), 627-649.

[47] D. Chinea, M. de León, J.C. Marrero: The constraint algorithm for time-dependent Lagrangians. *J. Math. Phys* ., **35** (1994), 3410-3447.

[48] M. McLean, L.K. Norris: Covariant Field Theory on Frame Bundles of Fibered Manifolds. *J. Math. Phys.* 41 (2000), 6808-6823.

[49] T. de Donder, "Théorie invariantive du calcul des variations, nouvelle édit.," Gauthier-Villars, Paris (1935); H. Weyl, "Geodesic fields in the calculus of variations for multiple integrals," Ann. Math. (2), **36**, 607-629 (1935).

[50] C. Carathéodory, "Über die Variationsrechnung bei mehrfachen Integralen," *Acta Szeged Sect. Scient. Mathem.*, **4**, 193-216 (1929).

[51] J. Kijowski: A finite-dimensional canonical formalism in the classical field theory. *Comm. Math. Phys.* **30** (1973), 99-128.

[52] J. Sniatycki: On the geometric structure of classical field theory in Lagrangian formulation. *Math. Proc. Cambridge Philos. Soc.* **68** (1970), 475-484.

[53] P.L. García, A. Pérez-Rendón: Symplectic approach to the theory of quantized fields, I. *Comm. Math. Phys.* **13** (1969) 24-44.

[54] P.L. García, P. L.; A. Pérez-Rendón: Symplectic approach to the theory of quantized fields, II. *Arch. Ratio. Mech. Anal.* **43** (1971), 101-124.

[55] G. Martin: Dynamical structures for k-vector fields. *Internat. J. Theoret. Phys.* **27** (1988), 571-585.

[56] G. Martin: A Darboux theorem for multi-symplectic manifolds. *Lett. Math. Phys.* 16 (1988), 133-138.

[57] M.J. Gotay: An exterior differential systems approach to the Cartan form. In: *Symplectic geometry and mathematical physics (Aix-en-Provence, 1990)*. Progr. Math., 99, Birkhäuser Boston, Boston, MA, 1991, pp. 160-188.

[58] M.J. Gotay: A multisymplectic framework for classical field theory and the calculus of variations, I. Covariant Hamiltonian formalism. In: *Mechanics, analysis and geometry: 200 years after Lagrange*. North-Holland Delta Ser., North-Holland, Amsterdam, 1991, pp. 203-235.

[59] M.J. Gotay: A multisymplectic framework for classical field theory and the calculus of variations, II. Space + time decomposition. *Differential Geom. App.* **1** (1991), 375-390.

[60] E. Binz, J. Sniatycki, H. Fischer: *Geometry of classical fields*. North-Holland Mathematics Studies, **154**. North-Holland Publishing Co., Amsterdam, 1988.

[61] A. Roux: *Jets et connexions*. Publ. Dep. de Mathématiques, Lyon, 1975.

[62] P. Libermann: Parallélismes. *J. Differential Geom.* **8** (1973), 511-539.

[63] G. Giachetta, L. Mangiarottti, G. Sardanashvily: *New Lagrangian and Hamiltonian Methods in Field Theory*. World Scientific, Singapore, 1997.

[64] G. Giachetta, L. Mangiarotti, G. Sardanashvily: Covariant Hamilton equations for field theory. *J. Phys. A: Math. Gen.* **32** (1999), 6629-6642.

[65] G. Giachetta, L. Mangiarotti, G. Sardanashvily: BRST-extended polysymplectic Hamiltonian formalism for field theory. *Nuovo Cimento B* (12) 114 (1999), 939-955.

[66] I.V. Kanatchikov: Novel algebraic structures from the polysymplectic form in field theory. In: *Physical Applications and Mathematical Aspects of Geometry, Groups and Algebra*, Vol. 2, H.A. Doebner, W. Scherer, C. Schulte eds., World Scientific, Singapore 1997, pp. 894-899.

[67] I. I.V. Kanatchikov: On field theoretic generalizations of a Poisson algebra. *Rep. Math. Phys.* **40** (1997), 4225-234.

[68] I. I.V. Kanatchikov: Canonical structure of Classical Field Theory in the polymomentum phase space. *Rep. Math. Phys.* **41** (1998), 49-90.

[69] G. Sardanishvily: *Gauge theory in jet manifolds*. Hadronic Press, Inc, Palm Harbor, 1993.

[70] G. Sardanashvily: *Generalized Hamiltonian formalism for field theory. Constraint systems*. World Scientific, Singapore, 1995.

[71] G. Sardanashvily: Stress-energy-momentum tensors in constraint field theories. *J. Math. Phys.* **38** (1997), 847-866.

[72] G. Sardanashvily: SUSY-extended field theory. *Internat. J. Modern Phys. A* **15** (2000), 3095-3112.

[73] G. Sardanashvily, O. Zakharov, On application of the Hamilton formalism in fibred manifolds to field theory. *Differential Geom. App.* **3** (1993), 245-263.

[74] R.O. Fulp, J. K. Lawson and L. K. Norris: Geometric prequantization on the spin bundle based on n-symplectic geometry: The Dirac equation. *Int. J. Theor. Phys.* **33** (1994) 1011-1028.

[75] A. Awane, M. Goze: *Pfaffian systems, k-symplectic systems*. Kluwer Academic Publishers, Dordrecht, 2000.

[76] A. Echevarría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Geometry of Lagrangian first-order classical field theories. *Fortschr. Phys.* **44** 3 (1996), 235-280.

[77] A. Echevarría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: Geometry of multisymplectic hamiltonian first-order field theories. *J. Math. Phys.* **41** (2000), 7402-7444.

[78] A. Echevarría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: A geometrical analysis of the field equations in field theory. *arXiv:math-ph/0105018*.

[79] S.P.Hrabak: On a Multisymplectic Formulation of the Classical BRST symmetry for First Order Field Theories Part I: Algebraic Structures. arXiv.org/math-ph/9901012

[80] S.P.Hrabak: On a Multisymplectic Formulation of the Classical BRST Symmetry for First Order Field Theories Part II: Geometric Structures. arXiv.org/math-ph/9901013

[81] M. de León, P.R. Rodrigues: Formalisme hamiltonien symplectique sur les fibrés tangents d'ordre supérieur. *C. R. Acad. Sci. Paris*, série II, **301**(1985), 103-106.

[82] M. de León, P.R. Rodrigues: *Generalized Classical Mechanics and Field Theory*. North-Holland Mathematics Studies, 112, North-Holland, Amsterdam, 1985.

[83] M. de León, P.R. Rodrigues: A contribution to the global formulation of the higher order Poincaré-Cartan form. *Lett. Math. Phy.* **14** 4 (1987), 353-362.

[84] M. de León, P.R. Rodrigues: n^k-Almost Tangent Structures and the Hamiltonization of Higher Order Field Theories. *J. Math. Phys.* **30** (1989), 1351-1353.

[85] L. Mangiarotti, G. Sardanashvily: *Connections in classical and quantum field theory*. World Scientific, Singapore .

[86] J.E. Marsden, S. Shkoller: Multisymplectic geometry, covariant hamiltonians and water wawes. *Math. Proc. Cambridge. Philos. Soc.* **125** (1999), 553-575.

[87] M. Puta: Some remarks on the k-symplectic manifolds. *Tensor, N.S.* **47** (1988), 109-115.

[88] D.J. Saunders: An alternative approach to the Cartan form in Lagrangian field theories. *J. Phys. A: Math. Gen.* **20** (1987), 339-349.

[89] D.J. Saunders: Jet fields, connections ans second order differential equations. *J. Phys. A: Math. Gen.* **20** (1987), 3261-3270.

[90] J. Sniatycki: The Cauchy data space formulation of classical field theory. *Rep. Math. Phys.* **19** (1984), 407-422.

[91] N. Woodhouse, *Geometric Quantization*, 2nd ed. (Oxford Press, Oford, 1992).

[92] K. Yano, S. Ishihara: Horizontal lifts from manifolds to its tangent bundle. *J. Math. and Mech.* **16** (1967), 1015-1030.

[93] K. Yano. S. Ishihara: *Tangent and cotangent bundles*. Marcel Dekker, New York, 1973.