Geometric Prequantization on the Spin Bundle Based on N-symplectic Geometry: The Dirac Equation*

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Abstract. We present preliminary results for a prequantization procedure that leads in a natural way to the Dirac equation. The starting point is the recently introduced n-symplectic geometry on the bundle of linear frames LM of an n-dimensional manifold M in which the $\mathbf{R}^n$-valued soldering 1-form θ on LM plays the role of the n-symplectic potential. On a 4-dimensional spacetime manifold we consider the tensorial $\mathbf{R}^4 \otimes \mathbf{R}^4$-valued function $\hat{g}$ on LM determined by the spacetime metric tensor $\vec{g}$ as the Hamiltonian for free observers and determine the associated $\mathbf{R}^4$-valued Hamiltonian vector field $\hat{X}_{\hat{g}} = X^i_{\hat{g}} \otimes r_i$. "Integration" of the $X^i_{\hat{g}}$ yields the dynamics of free observers on spacetime, namely parallel transport of linear frames along spacetime geodesics. In order to obtain a vector field on the spin bundle SM which is a lift of $\hat{X}_{\hat{g}}$ and which is induced by a vector field $\hat{X}_{\tilde{g}}$ for an appropriate mapping $\tilde{g}$, it is useful to define a prolongation $\widetilde{L^oM}$ of some bundle L^oM of oriented frames of M. If $GL^+(4,\mathbf{R})$ denotes the identity component of $GL(4,\mathbf{R})$ then $GL^+(4,\mathbf{R})$ is the structure group of L^oM and its double cover $\widetilde{GL^+(4,\mathbf{R})}$ is the structure group of $\widetilde{L^oM}$. We show that the lift $\tilde{\theta}$ of θ on L^oM to $\widetilde{L^oM}$ induces a natural 4-symplectic potential on $\widetilde{L^oM}$. If $\tilde{g}$ is the lift of $\hat{g}$ to $\widetilde{L^oM}$ then we find the $\mathbf{R}^4$-valued Hamiltonian vector field $\hat{X}_{\tilde{g}}$ on $\widetilde{L^oM}$ determined by $\tilde{g}$ and show that the vector fields $X^i_{\tilde{g}}$ on $\widetilde{L^oM}$ are tangent to the subbundle SM. "Integration" of the restriction of the $X^i_{\hat{g}}$ to SM now yields parallel transport of spin frames and thus tetrads along spacetime geodesics of $\vec{g}$. We consider a näive prequantization operator assignment $\hat{X}_{\tilde{g}} \mapsto \mathcal{P}_{\tilde{g}} := ih\gamma_i X^i_{\tilde{g}}$ acting on $\mathbf{C}^4$-spinors in the standard representation of $SL(2,\mathbf{C})$. The eigenvalue equation for the system of new Hilbert space operators yields the Dirac equation.

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1 Introduction

The Kostant-Souriau theory of geometric quantization [5, 11] that has been developed over the last 20 plus years takes symplectic geometry as the basic geometrical building block. That is to say, to set up the fundamental structure of the theory one does not need to assume a Riemannian metric tensor, a linear connection, or any other geometrical structure beyond symplectic geometry. These additional structures are only needed once one wants to work out the details of a specific model within the context of the general theory. Thus, for example, the theory of the free relativistic point particle in a curved spacetime $(M, \vec{g})$ follows from the general theory applied to the Hamiltonian $H = \tilde{g}$ thought of as the mass-squared operator. Here $\tilde{g}$ is the $\mathbf{R}$-valued function on T^*M determined by the metric tensor $\vec{g}$, and $\tilde{g}$ takes the quadratic form $\tilde{g} = g^{ij}(q)p_i p_j$ in local coordinates (q^i, p_j) on T^*M. In the Schrödinger representation one finds [10] the Klein–Gordon equation as the eigenvalue equation for the mass-squared operator. Thus the standard theory built on symplectic geometry on T^*M, the free particle Hamiltonian $H = \tilde{g}$ and the Schrödinger representation leads to the Klein–Gordon equation rather than the Dirac equation. On the other hand, in Souriau's derivation of the Dirac equation [11] he assumes the existence of a representation of Dirac 4-spinors and uses the symplectic structure on coadjoint orbits in the dual Poincaré Lie algebra $P(4)^*$. The new derivation of the Dirac equation presented in this chapter will follow from a different approach that is in a sense a mixture of these two approaches.

Our basic idea is to abandon standard symplectic geometry on T^*M and work instead with a generalized symplectic geometry on the bundle of linear frames LM. We shall also assume the existence of 4-spinor representations. The bundle of linear frames LM of an n-dimensional manifold M supports a natural structure, based on the $\mathbf{R}^n$-valued soldering one-form, that may be viewed as an "n-symplectic structure." We refer to the resulting geometry as "n-symplectic geometry". The allowable observables associated with this new geometry on LM contain the tensor-valued functions representing contravariant tensor fields on M. The metric tensor is the Hamiltonian tensor for free observers, thus serving as the analogue of the free particle Hamiltonian on T^*M. For a four-dimensional spacetime manifold M we determine by equation (3.1) the $\mathbf{R}^4$-valued Hamiltonian vector field, or four-dimensional Hamiltonian distribution which corresponds to the metric tensor. The classical dynamics associated with the Hamiltonian distribution is the parallel transport of frames along geodesics in M. Note that in 4-symplectic geometry the free observer Hamiltonian yields four Hamiltonian vector fields compared with the single Hamiltonian vector field defined on T^*M. The fact that we obtain four Hamiltonian vector fields clearly suggests the possibility of constructing the Dirac operator via a geometric quantization approach.

To lift this Hamiltonian distribution to SM, the spin bundle over M, we must first restrict it to O^oM, a component of the orthonormal frame bundle $(OM, \vec{g})$, with group the identity component of the Lorentz group [1, p. 81]. However, if we define $\hat{g}$ to be the $\mathbf{R}^4 \otimes \mathbf{R}^4$-valued tensorial 0-form on LM determined by $\vec{g}$, then $\hat{g} \mid O^oM$ is constant, corresponding to the view of O^oM as a constant energy-momentum surface in LM. Hence $d(\hat{g} \,|\, O^oM) \equiv 0$ and we cannot determine Hamiltonian vector fields. So in order to determine the Hamiltonian distribution $\hat{X}_{\hat{g}}$ on O^oM first we must work on LM, and then restrict to O^oM. Similarly in order to find a Hamiltonian distribution on SM we must find a way to extend SM. It is the purpose of this chapter to demonstrate a technique which permits us to do so and consequently to obtain a Hamiltonian distribution on SM describing free observers. The resulting distribution then may be used to construct the Dirac equation.

Let L^oM be a subbundle of LM with group $GL^+(4,\mathbf{R})$, the identity component of $GL(4,\mathbf{R})$. Consider the bundle prolongation of L^oM to a $\widetilde{GL^+(4,\mathbf{R})}$ bundle, $\widetilde{L^oM}$. On $\widetilde{L^oM}$ we find the $\mathbf{R}^4$-valued Hamiltonian vector field $\hat{X}_{\tilde{g}}$ determined by the free particle Hamiltonian tensor $H = \tilde{g}$, where $\tilde{g}$ is the lift of $\hat{g}$ to $\widetilde{L^oM}$. Then we observe that "integration" of the system of vector fields yields SM as a subbundle of $\widetilde{L^oM}$ with structure subgroup $SL(2,\mathbf{C})$. A Hermitian operator, the naive prequantization operator, is defined on the restriction to SM of the Hamiltonian vector field on $\widetilde{L^oM}$. We may now assign to this Hermitian operator a representation as an operator on $L_2(SM,\mathbf{C}^4)$, the space of Dirac 4-spinors. As an end result we show that the Dirac equation arises as the eigenvalue equation for this spinor representation of the naive prequantization operator determined by the free particle Hamiltonian tensor H.

2 Survey of n-symplectic geometry on LM

Let M be an n-dimensional manifold and let LM be the principal fiber bundle of linear frames of M. The dimension of LM is the even number $n(n+1)$. A point $u \in LM$ will be denoted by the pair (p, e_i) where $p \in M$ and $(e_i) \equiv (e_1, e_2, \ldots, e_n)$ denotes a linear frame at p. The projection map $\pi : LM \to M$ is defined by $\pi(p, e_i) = p$. The structure group of LM is the general linear group $GL(n,\mathbf{R})$, which acts freely on the right of LM by $R_g(p, e_i) \equiv (p, e_i)\cdot g = (p, e_j g^j_i)$ for $g = (g^i_j) \in GL(n,R)$. The summation convention on repeated indices is employed throughout this chapter.

Local coordinates on LM may be defined as follows [8]. If (U, x^i) is a chart on M, then define local coordinates $(x^i, \pi^j_k) : \pi^{-1}(U) \to \mathbf{R}^n \times \mathbf{R}^{n^2}$ by $x^i(p, e_j) = x^i(p)$ and $\pi^j_k(p, e_i) = e^j(\frac{\partial}{\partial x^k})$. Here (e^j) , $j = 1, 2, \ldots, n$ denotes the coframe dual to the linear frame (e_j). Note that we follow the standard practice of using x^i to denote coordinates on both $U \subset M$ and $\pi^{-1}(U) \subset LM$.

The structure of LM is special in the sense that it supports a globally defined $\mathbf{R}^n$-valued one-form, the soldering one-form $\hat{\theta} = \theta^i r_i$. Here $r_1, r_2, \ldots, r_n$ denotes the standard basis of $\mathbf{R}^n$. Each point $u \in LM$ can be defined [4] as a linear isomorphism $u : \mathbf{R}^n \to T_{\pi(u)}M$. In local coordinates u can be defined by

$$u(\xi^i r_i) \equiv (p, e_j)(\xi^i r_i) := \xi^i e_i \quad , \tag{2.1}$$

with inverse

$$u^{-1}(X) \equiv (p, e_i)^{-1}(X) = e^i(X) r_i \quad , \quad X \in T_pM \ . \tag{2.2}$$

Then the soldering one-form θ may be defined by [4]

$$\theta(Y) \stackrel{def}{=} u^{-1}(\pi_* Y) \quad , \quad \forall\, Y \in T_u LM \ . \tag{2.3}$$

In local coordinates (x^i, π^j_k) the soldering one-form has the local expression

$$\theta^i r_i = (\pi^i_j dx^j) r_i \equiv (\pi^i_j r_i) dx^j \ . \tag{2.4}$$

Compare this form to the expression $\vartheta = p_j dx^j$ for the canonical one-form on T^*M in canonical coordinates. The difference is that the momentum coordinates p_j on T^*M are $\mathbf{R}$-valued while the $\hat{\pi}_j := \pi^i_j r_i$ on LM are $\mathbf{R}^n$-valued.

Consider now the exact $\mathbf{R}^n$-valued two-form $d\theta$. By (2.4) it has the local coordinate expression

$$d\theta = d\theta^i r_i = (d\pi^i_j \wedge dx^j) r_i \ . \tag{2.5}$$

Using equation (2.5), it is easy to show that $d\theta$ is nondegenerate in the sense that

$$X \rfloor \ d\theta = 0 \Leftrightarrow X = 0 \ . \tag{2.6}$$

Thus $d\theta$ has the basic properties of a symplectic structure, although it is $\mathbf{R}^n$-valued. This motivates the following definition:

Definition Let P be a principal fiber bundle over a manifold M with group G. Let the dimension of M be n. An *n-symplectic structure* on P is an $\mathbf{R}^n$-valued two-form ω on P that is closed and nondegenerate in the sense of equation (2.6). The pair (P, ω) is an *n-symplectic manifold.*

The theory of n-symplectic geometry on $(LM, d\theta)$ [8] is based on generalizing the basic structure equation

$$df = -X_f \rfloor \ d\vartheta \tag{2.7}$$

on T^*M to $(LM, d\theta)$. In (2.7) f denotes any smooth $\mathbf{R}$-valued function on T^*M . Since $d\theta$ is $\mathbf{R}^n$-valued the range of the variables changes in n-symplectic geometry. The simplest generalization of (2.7) is

$$d\hat{f} = -X_{\hat{f}} \rfloor \ d\theta \tag{2.8}$$

where now $\hat{f}$ is a smooth $\mathbf{R}^n$-valued function on LM. Since $d\theta$ is nondegenerate if a vector field $X_{\hat{f}}$ satisfies (2.8) for a given $\mathbf{R}^n$-valued function $\hat{f}$, then it will be unique. On the other hand, the soldering one-form θ transforms tensorially under right translations R_g for $g \in GL(n)$ according to $R^*_g \theta = g^{-1} \cdot \theta$. A consequence of this tensorial nature of θ is that not every $\mathbf{R}^n$-valued function on LM is compatible with equation (2.8). On the other hand all smooth $\mathbf{R}$-valued functions on T^*M are compatible with equation (2.7).

Let T^1 denote the set of $\mathbf{R}^n$-valued functions $\hat{f}$ on LM that transform tensorially under right translation by $R^*_g \hat{f} = g^{-1} \cdot \hat{f}$. Such functions are in one–one correspondence with vector fields on M. Denote by HF^1 the set of $\mathbf{R}^n$-valued functions on LM that are compatible with (2.8). Norris [8] shows that

$$HF^1 = T^1 \oplus C^\infty(M, \mathbf{R}^n) \tag{2.9}$$

where the second factor denotes the smooth $\mathbf{R}^n$-valued functions on LM that are invariant on fibers. For each $\hat{f} \in HF^1$ equation (2.8) assigns a unique Hamiltonian vector field $X_{\hat{f}}$. The Poisson bracket of $\hat{f}, \hat{g} \in HF^1$ is defined by

$$\{\hat{f}, \hat{g}\} = X_{\hat{f}}(\hat{g}) \tag{2.10}$$

and HF^1 is a Lie algebra under this bracket. Denote by HV^1 the set of Hamiltonian vector fields $X_{\hat{f}}$ determined by elements of HF^1. Then one shows that

$$[X_{\hat{f}}, X_{\hat{g}}] = X_{\{\hat{f},\hat{g}\}} \tag{2.11}$$

so that HV^1 forms a Lie algebra.

¿From (2.8) it is clear that the constant $\mathbf{R}^n$-valued functions in $C^\infty(M, \mathbf{R}^n) \subset HF^1$ are all mapped to the zero vector field. Identifying these constant functions with $\mathbf{R}^n$, we have that as Lie algebras

$$HV^1 = HF^1/\,\mathbf{R}^n \quad . \tag{2.12}$$

Strictly speaking, the bracket defined in (2.10) is not a Poisson bracket but simply a Lie bracket since it is not a Lie derivation. However, the bracket becomes a true Poisson bracket when HF^1 is combined with the higher rank symmetric $T^p\mathbf{R}^n$-valued observables. Let

$$ST^p = \{\hat{f} : LM \to \otimes_s^p \mathbf{R}^n | \hat{f}(u \cdot h) = h^{-1} \cdot \hat{f}(u) \ \ \forall\, h \in GL(n)\}$$

denote the vector space of functions on LM which have their values in the vector space $\otimes_s^p \mathbf{R}^n$, where $\otimes_s$ denotes the symmetric tensor product. Let $\mathrm{S}\mathcal{X}^p$ denote the vector space of symmetric rank p contravariant tensor fields on M and observe that each element of ST^p corresponds to a unique element of $\mathrm{S}\mathcal{X}^p$. We denote by $ST = \sum_{p=1}^{\infty} ST^p$ the infinite dimensional vector space which is the direct sum of the vector spaces ST^p.

An element $\hat{f} \in ST^p$ determines [8] an equivalence class $[\![X_{\hat{f}}]\!]$ of $\binom{n+p-2}{p-1}$ vector fields $[\![X_{\hat{f}}]\!]^{i_1 \ldots i_{p-1}}$ via the n-symplectic structure equation

$$d\hat{f}^{i_1 \ldots i_p} = -p!\, X_{\hat{f}}^{(i_1 \ldots i_{p-1}} \rfloor \ d\theta^{i_p)} \tag{2.13}$$

where round brackets on indices denotes symmetrization. We note that although $d\theta$ is nondegenerate in the sense of (2.6), because of the symmetrization in (2.13) the nondegeneracy is lost. For a given $\hat{f} \in ST^p$ equation (2.13) only determines the vector fields $X_{\hat{f}}^{i_1 \ldots i_{p-1}}$ up to addition of vector fields $Y^{i_1 \ldots i_{p-1}}$ satisfying the kernel equation

$$Y^{(i_1 \ldots i_{p-1}} \rfloor \ d\theta^{i_p)} = 0 \quad . \tag{2.14}$$

If a set of vector fields $Y^{i_1 \ldots i_{p-1}}$ satisfies (2.14), then each vector field $Y^{i_1 \ldots i_{p-1}}$ must be vertical. For a given $\hat{f} \in ST^p$ equation (2.13) thus determines an equivalence class of $T^p\mathbf{R}^n$-valued Hamiltonian vector fields $([\![X_{\hat{f}}]\!]^{i_1 \ldots i_{p-1}})$, where two $T^p\mathbf{R}^n$-valued vector fields are equivalent if their difference satisfies equation (2.14).

An element

$$\hat{f} = \hat{f}^{i_1 i_2 \dots i_p} r_{i_1} \otimes_s r_{i_2} \otimes_s \cdots \otimes_s r_{i_p} \in ST^p$$

has the local canonical coordinate representation

$$\hat{f}^{i_1 i_2 \dots i_p} = f^{j_1 j_2 \dots j_p}(x) \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_p}_{j_p} \quad . \tag{2.15}$$

The associated equivalence classes of Hamiltonian vector fields $[\![X_{\hat{f}}]\!]^{i_1 i_2 \dots i_{p-1}}$ determined by equation (2.13) have the local coordinate representations

$$\begin{aligned} X_{\hat{f}}{}^{i_1 i_2 \dots i_{p-1}} &= \frac{1}{(p-1)!} f^{j_1 j_2 \dots j_{p-1} k} \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_{p-1}}_{j_{p-1}} \frac{\partial}{\partial x^k} \\ &\quad - \frac{1}{p!} \left(\frac{\partial f^{j_1 j_2 \dots j_p}}{\partial x^a} \pi^{i_1}_{j_1} \pi^{i_2}_{j_2} \cdots \pi^{i_{p-1}}_{j_{p-1}} \pi^b_{j_p} + T_a^{i_1 i_2 \dots i_{p-1} b} \right) \frac{\partial}{\partial \pi^b_a} \end{aligned} \tag{2.16}$$

where the components $T_a^{i_1 i_2 \dots i_{p-1} b}$ must satisfy

$$T_a^{(i_1 i_2 \dots i_{p-1} b)} = 0 \tag{2.17}$$

but are otherwise arbitrary.

The fact that one obtains equivalence classes of vector fields rather than vector fields for the higher rank observables does not interfere with the basic algebraic structures in n-symplectic geometry. For each $p \geq 1$ the set of equivalence classes of $\otimes_s^{p-1} \mathbf{R}^n$-valued vector fields on LM, with equivalence defined as above, forms an infinite-dimensional vector space. Denote by $HV(ST^p)$ the vector space of $\otimes_s^{p-1} \mathbf{R}^n$-valued equivalence classes of vector fields determined by elements of ST^p by equation (2.13). For $\hat{f} \in ST^p$ and $\hat{g} \in ST^q$ define the Poisson bracket $\{ \ , \ \} : ST^p \times ST^q \to ST^{p+q-1}$ by

$$\{\hat{f}, \hat{g}\}^{i_1 i_2 \dots i_{p+q-1}} = p! X_{\hat{f}}{}^{(i_1 i_2 \dots i_{p-1}} \left(\hat{g}^{i_p i_{p+1} \dots i_{p+q-1})} \right) \tag{2.18}$$

where $X_{\hat{f}}{}^{i_1 i_2 \dots i_{p-1}}$ is any representative of the equivalence class $[\![X_{\hat{f}}]\!]^{i_1 i_2 \dots i_{p-1}}$. The bracket so defined is easily shown to be independent of the choice of representatives and has all the properties of a Poisson bracket. In fact when the bracket defined here is reexpressed on the base manifold M, it gives [8] the differential concomitant of Schouten [9] and Nijenhuis [7] of the symmetric tensor fields corresponding to $\hat{f}$ and $\hat{g}$.

Theorem 2.1 *The space ST of symmetric tensorial functions on LM is a Poisson algebra with respect to the Poisson bracket defined in (2.18).*

It is convenient to introduce the multi-index notation

$$r_{i_1 i_2 \dots i_{p-k}} \equiv r_{i_1} \otimes_s r_{i_2} \otimes_s \cdots \otimes_s r_{i_{p-k}} \text{ for } 0 \leq k \leq p-1.$$

Let

$$[\![\hat{X}_{\hat{f}}]\!] = [\![X_{\hat{f}}]\!]^{i_1 i_2 \ldots i_{p-1}} r_{i_1 i_2 \ldots i_{p-1}} \text{ and } [\![\hat{X}_{\hat{g}}]\!] = [\![X_{\hat{g}}]\!]^{i_1 i_2 \ldots i_{p-1}} r_{i_1 i_2 \ldots i_{p-1}}$$

denote the vector valued equivalence classes of vector fields determined by $\hat{f} \in ST^p$ and $\hat{g} \in ST^q$. Define a bracket by

$$\begin{aligned} [[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] &= [[\![X_{\hat{f}}]\!]^{i_1 i_2 \ldots i_{p-1}} \ , \ [\![X_{\hat{g}}]\!]^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}} \\ &= [\![[X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}} \ , \ X_{\hat{g}}{}^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}}]\!] \end{aligned} \tag{2.19}$$

where the bracket on the right-hand side is the ordinary Lie bracket of vector fields calculated using arbitrary representatives. One shows that

$$[X_{\hat{f}}{}^{i_1 i_2 \ldots i_{p-1}} \ , \ X_{\hat{g}}{}^{i_p i_{p+1} \ldots i_{p+q-2}}] r_{i_1 i_2 \ldots i_{p+q-2}} \in [\![X_{\{\hat{f},\hat{g}\}}]\!] \ , \tag{2.20}$$

and thus the bracket defined in (2.19) is well-defined, and we write

$$[[\![\hat{X}_{\hat{f}}]\!], [\![\hat{X}_{\hat{g}}]\!]] = [\![\hat{X}_{\{\hat{f},\hat{g}\}}]\!] \ . \tag{2.21}$$

Moreover, the bracket defined in (2.19) is anti-symmetric. Denote the direct sum of the vector spaces $HV(ST^p)$ by $HV(ST)$.

Theorem 2.2 *The vector space $HV(ST)$ of vector-valued equivalence classes of Hamiltonian vector fields on LM is a Lie algebra with respect to the bracket defined in (2.19).*

In general, it can be observed that n-symplectic geometry selects "allowable observables" in the sense that not every $\otimes^p_s \mathbf{R}^n$-valued function on LM is compatible with (2.13). It is known [8] that the most general $\otimes^p_s \mathbf{R}^n$-valued function on LM that can satisfy (2.13) for some set of vector fields must be a polynomial in the momentum coordinates with coefficients in the set of functions that are invariant on fibers on LM. We denote this set by SHF^p. For a given $p \geq 1$ the homogeneous degree p polynomials in SHF^p form the set ST^p, while for $p > 2$ the lower-degree polynomials do not in general correspond to elements of ST^q for $0 \leq q < p$. The reader is referred to Norris [8] for more details.

3 The metric tensor as a generalized Hamiltonian tensor for free inertial observers

Now we wish to study the dynamics generated by the spacetime metric tensor Hamiltonian g on LM, where M is a 4-dimensional spacetime manifold, within the context of 4-symplectic geometry. First we consider the classical phenomena and then we examine the prequantization assignments.

Let $\vec{g} = g^{ij}\partial_i \otimes \partial_j$ be the local coordinate form of the metric tensor on spacetime, and let $\hat{g} = g^{ij}\pi^a_i\pi^b_j r_a \otimes_s r_b$ denote the corresponding symmetric tensorial function on LM. As a special case of (2.13), the 4-symplectic structure equation is

$$d\hat{g}^{ij} = -2X^{(i}_{\hat{g}} \rfloor \; d\theta^{j)} \tag{3.1}$$

and the associated Hamiltonian vector fields $X^i_{\hat{g}}$ determined by this equation have the local expressions

$$X^i_{\hat{g}} = g^{ab}\pi^i_a \frac{\partial}{\partial x^b} - \frac{1}{2}\{\frac{\partial g^{ab}}{\partial x^j}\pi^i_a\pi^k_b + T^{ik}_j\}\frac{\partial}{\partial \pi^k_j} \; . \tag{3.2}$$

To fix the nonuniqueness globally in these vector fields one can impose the invariantly defined constraint equation

$$X^i_{\hat{g}} \rfloor \; X^j_{\hat{g}} \rfloor \; d\theta^k = 0 \quad \forall \; i,j,k = 0,\ldots,3 \; . \tag{3.3}$$

This condition uniquely determines the arbitrary functions T^{ij}_k so that the resulting Hamiltonian vector fields are

$$X^i_{\hat{g}} = g^{ab}\pi^i_a \frac{\partial}{\partial x^b} + \Gamma^b_{jc} g^{ac}\pi^i_a\pi^k_b \frac{\partial}{\partial \pi^k_j} \; . \tag{3.4}$$

The functions Γ^b_{jc} are the Christoffel symbols of the Levi-Civita connection defined by $\vec{g}$.

It is straightforward to check that the distribution on LM spanned by the vector fields

$$\begin{aligned} B_k &= \hat{g}_{ki} X^i_{\hat{g}} \\ &= (\pi^{-1})^j_k(\frac{\partial}{\partial x^j} + \Gamma^a_{ij}\pi^b_a \frac{\partial}{\partial \pi^b_i}) \end{aligned} \tag{3.5}$$

is the horizontal distribution of the Levi-Civita connection. Here $\hat{g}_{ij} = (g_{ab})(\pi^{-1})^a_i(\pi^{-1})^b_j$ where the functions g_{ij} are the components of the matrix inverse of (g^{ij}) and $(\pi^{-1})^j_k\pi^k_i = \delta^j_i$. The vector fields B_i are easily seen to be the "standard horizontal vector fields" [4] determined by the connection. Note that the constraint equation (3.3) is equivalent to the property that the Levi-Civita connection has no torsion.

Let us next consider the dynamics associated with the Hamiltonian vector fields $X^i_{\hat{g}}$ defined in equation (3.4). When there is only a single Hamiltonian vector field $X_{\hat{f}}$, as in the case for the tensorial $\mathbf{R}^4$-valued function $\hat{f}$ as well as in standard symplectic geometry on T^*M, then the dynamics is given by the integral curves of $X_{\hat{f}}$. One can ask if the distribution spanned by the $X^i_{\hat{g}}$ is integrable, but it is well known that only flat connections have integrable distributions. On the other hand, the vector fields B_k, and hence also the vector fields $X^i_{\hat{g}}$, are tangent to the subbundle of orthonormal linear frames OM determined by $\hat{g}$. Thus we may define an "integral" of the set of Hamiltonian vector fields $X^i_{\hat{g}}$ to be OM [8]. Because the integrals of the $X^i_{\hat{g}}$ do not form an involutive distribution, the subbundle OM is not a leaf of a foliation induced by the $X^i_{\hat{g}}$ alone. Instead OM serves as the analogue of the "constant-energy surfaces" in standard symplectic geometry.

Since a section of OM represents a local orthonormal linear frame field on M we may conclude that *the dynamics defined by the four Hamiltonian vector fields is the dynamics of orthonormal frames, and hence the dynamics of local observers on spacetime.* More explicitly, consider the integral curves of the "timelike" Hamiltonian vector field $X^0_{\hat{g}}$. (Here "timelike" means that $X^0_{\hat{g}}$ projects to a timelike vector field on M.) The differential equations for the integral curve of $X^0_{\hat{g}}$ are, from equation (3.4),

$$\frac{dx^i}{dt} = g^{ij}\pi^0_j \ , \qquad\qquad \frac{d\pi^k_j}{dt} = \Gamma^b_{jc} g^{ac} \pi^0_a \pi^k_b \ . \tag{3.6}$$

These equations decouple into the two sets of equations. For $k = 0$ we obtain

$$\frac{dx^i}{dt} = g^{ij}\pi^0_j \ , \qquad\qquad \frac{d\pi^0_j}{dt} = \Gamma^b_{jc} g^{ac} \pi^0_a \pi^0_b \ , \tag{3.7}$$

and for $k = \alpha = 1, 2, 3$,

$$\frac{d\pi^\alpha_j}{dt} = \Gamma^b_{jc} g^{ac} \pi^0_a \pi^\alpha_b \ . \tag{3.8}$$

The pair of equations (3.7) combine into the second order geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \ , \tag{3.9}$$

while equation (3.8) can be rewritten as

$$\frac{D\pi^\alpha_j}{Dt} = \frac{d\pi^\alpha_j}{dt} - \Gamma^k_{ij}\frac{dx^i}{dt}\pi^\alpha_k = 0 \ , \quad \alpha = 1, 2, 3 \ . \tag{3.10}$$

These last equations are just the equations for parallel transport of the 1, 2, and 3 legs of a coframe along the geodesic determined by equation (3.9). The result is that $X^0_{\hat{g}}$ *generates parallel transport of linear frames and coframes along timelike geodesics of* $\hat{g}$. If we repeat this discussion for, say $X^1_{\hat{g}}$, then again we obtain parallel transport of linear frames along geodesics, but these geodesics will generally be spacelike. The four Hamiltonian vector fields X^i_g associated with the spacetime metric tensor can therefore be used to construct the local Lorentzian coordinate systems carried by a freely-falling observer.

4 Development of 4-symplectic geometry on the orthonormal frame bundle and the spin bundle of spacetime.

Let M be a four-dimensional manifold. The orthonormal frame bundle OM can be obtained as a bundle reduction of LM via symmetry breaking by the tensor field $\hat{g} : LM \to T^2_0\,\mathbf{R}^4$, specifically, $OM = \hat{g}^{-1}(\eta)$ where $\eta = \mathrm{diag}(1, -1, -1, -1)$. The reduced bundle OM has as its standard fiber the Lorentz group $O(1, 3)$. Let M be space and time orientable and denote an arbitrary choice of component of OM by O^oM. Then O^oM is a principal fiber bundle over M with standard fiber, $SO^0(1,3)$, the connected identity component of the Lorentz group.

Now recall the structure of the spin bundle SM over O^oM [1]. SM is a principal fiber bundle over M with standard fiber $SL(2,\mathbf{C})$, the special linear group on $\mathbf{C}^2$. The spin structure consists of SM with a bundle homomorphism $\lambda : SM \to O^oM$ and universal covering group homomorphism $\Lambda : SL(2,\mathbf{C}) \to SO^0(1,3)$ such that $\lambda(p \cdot g) = \lambda(p) \cdot \Lambda(g)$ for $p \in SM$ and $g \in SL(2,\mathbf{C})$. SM is then a double cover of O^oM. We recall that if M is noncompact then such a spin structure exists if and only if O^oM is trivial [3].

Motivated by the fact that the bundle of frames of 2-spinors is such a spin structure over O^oM, we would like to put a 4-symplectic structure on an arbitrary spin bundle. A natural choice would be a 4-symplectic structure inherited from O^oM. We may simply pull back the canonical one-form on O^oM. Let $\tilde{\theta} := \lambda^*(\theta \,\big|\, T(O^oM))$. Then $d\tilde{\theta} = \lambda^*(d\theta \,\big|\, T(O^oM))$. For $g \in SL(2,\mathbf{C})$ it follows that $R_g^*\tilde{\theta} = \Lambda(g)^{-1} \cdot \tilde{\theta}$. So $d\tilde{\theta}$ is tensorial, closed, and nondegenerate in the sense of (2.6) and thus is a candidate for an $\mathbf{R}^4$-valued 4-symplectic form on SM.

Now for the tensorial metric function $\hat{g} : LM \to T^2_0\,\mathbf{R}^4$ [8], we note that $\hat{g} \,\big|\, O^oM = \eta$ and $d(\hat{g} \,|\, O^oM) = 0$. Thus the 4-symplectic equation becomes

$$X_\eta^{(i} \lrcorner \; d\theta^{j)} = 0. \tag{4.1}$$

Observe that the constraint equation (3.2) is identically satisfied in this case. We have for Hamiltonian vector fields on O^oM,

$$X_\eta^i = T_j^{ik}\frac{\partial}{\partial \pi_j^k}; \qquad \text{where } T_j^{(ik)} = 0. \tag{4.2}$$

Consequently the corresponding Hamiltonian vector fields at $u \in SM$ induced by isomorphism $\lambda_* \,|\, T_uSM$ are also purely vertical. So this construction of Hamiltonian vector fields on SM is too restrictive.

5 Prolongation of frame bundles and lifts of metric connection geometry

Recognizing that the restriction of the 4-symplectic equation to O^oM is the source of our difficulty in obtaining Hamiltonian vector fields, we are motivated to enlarge SM to recover the geometry of the full frame bundle. We do so by the method of prolongation of the frame bundle [2], that is, the construction of a principal fiber bundle whose structure group is the universal cover of the structure group of the frame bundle. Note that L^oM, the connected component of LM containing O^oM as a subbundle, can be reconstructed from O^oM as it can be identified with the extension $O^oM \times_{SO^0(1,3)} GL^+(4,\mathbf{R})$, where $GL^+(4,\mathbf{R})$ is the component of $GL(4,\mathbf{R})$ containing the identity. We recall that the "canonical identification" [2]

$$O^oM \times_{SO^0(1,3)} GL^+(4,\mathbf{R}) \to L^oM : [(p,e_i),g] \mapsto (p, e_i \cdot g)$$

is a bundle isomorphism. Although L^oM is a principal fiber bundle with fiber $GL^+(4,\mathbf{R})$, it is also a bundle associated to the principal fiber bundle O^oM.

To motivate the analogous extension of SM, let us study the spin structure map $\lambda : SM \to O^oM$. The group $SL(2,\mathbf{C})$ is the universal cover of $SO^0(1,3)$ and λ is a double cover respecting the covering homomorphism. Denote the universal cover of $GL^+(4,\mathbf{R})$ as $\widetilde{GL^+(4,\mathbf{R})}$ with projection $\tilde{\Lambda} : \widetilde{GL^+(4,\mathbf{R})} \to GL^+(4,\mathbf{R})$. We observe that $\widetilde{GL^+(4,\mathbf{R})}$ is a 2–1 cover of $GL^+(4,\mathbf{R})$.

Now $SL(2,\mathbf{C})$ is a proper Lie subgroup of $\widetilde{GL^+(4,\mathbf{R})}$. Indeed it is known [6] that $SL(2,\mathbf{C})$ is isomorphic to the simply connected spin group $Spin^o(1,3)$, and thus is generated by units in the Clifford algebra $Cl(1,3) = Cl(\mathbf{R}^4,\eta)$. Moreover $Cl(1,3)$ is a subalgebra of $Cl(3,3)$ whose units in turn generate $Spin^o(3,3) \simeq \widetilde{SL(4,\mathbf{R})}$, the universal covering group of $SL(4,\mathbf{R})$. It follows that $SL(2,\mathbf{C})$ is a Lie subgroup of $\widetilde{SL(4,\mathbf{R})}$. Now $GL^+(4,\mathbf{R}) \simeq \mathbf{R}^+ \times SL(4,\mathbf{R})$ and $\mathbf{R}^+$ is contractable, so $\widetilde{GL^+(4,\mathbf{R})} \simeq \mathbf{R}^+ \times \widetilde{SL(4,\mathbf{R})}$ and thus we have that $\widetilde{SL(4,\mathbf{R})}$ is a Lie subgroup of $\widetilde{GL^+(4,\mathbf{R})}$. Thus we obtain the following short exact sequences:

$$\begin{array}{ccccccccc} <1> & \to & Z_2 & \to & \widetilde{GL^+(4,\mathbf{R})} & \overset{\tilde{\Lambda}}{\to} & GL^+(4,\mathbf{R}) & \to & <1> \\ & & & & \tilde{\jmath}\uparrow & & j\uparrow & & \\ <1> & \to & Z_2 & \to & SL(2,\mathbf{C}) & \overset{\Lambda}{\to} & SO^0(1,3) & \to & <1> \end{array} \tag{5.1}$$

where $\tilde{\jmath}$ and j are the inclusion homomorphisms.

Now define $\widetilde{L^oM} = SM \times_{SL(2,\mathbf{C})} \widetilde{GL^+(4,\mathbf{R})}$ as a prolongation of SM. Then $\widetilde{L^oM}$ is a principal fiber bundle over M with standard fiber $\widetilde{GL^+(4,\mathbf{R})}$ and with action defined by $[u,a]\cdot b = [u,ab]$ for $u \in SM$ and $a,b \in \widetilde{GL^+(4,\mathbf{R})}$. Observe that SM may be identified as a submanifold of $\widetilde{L^oM}$ since $SM = SM \times_{SL(2,\mathbf{C})} SL(2,\mathbf{C})$. Define a map $\tilde{\lambda} : \widetilde{L^oM} \to L^oM : [u,a] \mapsto [\lambda(u), \tilde{\Lambda}(a)]$. Then $\tilde{\lambda}$ is well-defined and the following diagram commutes.

$$\begin{array}{ccc} SM & \overset{\tilde{\imath}}{\hookrightarrow} & \widetilde{L^oM} \\ \lambda\downarrow & & \tilde{\lambda}\downarrow \\ O^oM & \overset{i}{\hookrightarrow} & L^oM \end{array} \tag{5.2}$$

Moreover,

$$\tilde{\lambda}([u,a]\cdot b) = \tilde{\lambda}([u,a])\cdot\tilde{\Lambda}(b) \;\forall u \in SM,\; \forall a,b \in \widetilde{GL^+(4,\mathbf{R})}.$$

So $(\widetilde{L^oM}, \tilde{\lambda})$ is a prolongation of L^oM. The topological condition for existence of the prolongation of L^oM is the same as the condition for the existence of a spin structure [2].

Note that SM can be recovered from $\widetilde{L^oM}$ by symmetry breaking via the tensor field $\tilde{g} : \widetilde{L^oM} \to T^2_0\mathbf{R}^4$, where we define $\tilde{g} := \tilde{\lambda}^*\hat{g}$. As before, $\hat{g} : L^oM \to T^2_0\mathbf{R}^4$ is a tensorial function corresponding to a Riemannian metric tensor field on M. $\tilde{g}$ is also a tensorial map and $SM = \tilde{g}^{-1}(\eta)$. Indeed recall that O^oM is obtained by symmetry breaking of L^oM, i.e., $O^oM = \hat{g}^{-1}(\eta)$ where $\eta = \mathrm{diag}(1,-1,-1,-1)$. From the construction of prolongations and extensions in diagram (5.2) we have that

$$\tilde{g}^{-1}(\eta) = \tilde{\lambda}^{-1}\circ\hat{g}^{-1}(\eta) = \tilde{\lambda}^{-1}(O^oM) = SM. \tag{5.3}$$

Consider now the problem of lifting connections through this diagram. Let ω_g be the unique Levi-Civita connection on O^oM determined by the metric function $\hat{g}$. The one-form ω_g extends naturally to a torsionless metric connection $\bar{\omega}_g$ on L^oM, considered to be the unique Levi-Civita connection on L^oM. Now lift the canonical one-form to SM, defining as before, $\tilde{\theta} := \lambda^*\theta$. Observe that $\tilde{\theta}$ is tensorial relative to the representation $SL(2,\mathbf{C}) \to GL(4,\mathbf{R})$ given by $a \cdot v = \Lambda(a)(v)$ for $a \in SL(2,\mathbf{C})$ and $v \in \mathbf{R}^4$ and it also vanishes on vertical vectors [1]. Define $\tilde{\omega}_g := \Lambda_*^{-1} \circ (\lambda^*\omega_g)$. Then $\tilde{\omega}_g$ is a connection on SM [1], sometimes called the spin connection. Furthermore $\tilde{\omega}_g$ is torsionless in the sense that $D^{\tilde{\omega}_g}\tilde{\theta} = 0$, where $D^{\tilde{\omega}_g}$ is covariant differentiation with respect to connection $\tilde{\omega}_g$. To verify this, first note the above representation induces a Lie algebra representation $sl(2,\mathbf{C}) \to gl(4,\mathbf{R})$ given by $A \cdot v = \Lambda_*(A)(v)$ for $A \in sl(2,\mathbf{C}),\ v \in \mathbf{R}^4$. Then observe that

$$\begin{aligned} D^{\tilde{\omega}_g}\tilde{\theta} &= d\tilde{\theta} + \tilde{\omega}_g \dot{\wedge} \tilde{\theta} \\ &= \lambda^* d\theta + (\Lambda_* \circ \Lambda_*^{-1} \circ \lambda^*\omega_g) \wedge (\lambda^*\theta) \\ &= \lambda^*(D^{\omega_g}\theta) \\ &= 0 \end{aligned} \tag{5.4}$$

where

$$\tilde{\omega}_g \dot{\wedge} \tilde{\theta}(X,Y) = \frac{1}{2}[\tilde{\omega}_g(X) \cdot \tilde{\theta}(Y) - \tilde{\omega}_g(Y) \cdot \tilde{\theta}(X)], \qquad \forall X, Y \in T_uSM.$$

A question arises. Is this the same connection we would obtain by lifting $\bar{\omega}_g$ to $\widetilde{L^oM}$ and restricting the result to SM? Define

$$\tilde{\bar{\omega}}_g := \tilde{\Lambda}_*^{-1} \circ \tilde{\lambda}^* \bar{\omega}_g.$$

Then it follows that

$$\tilde{\bar{\omega}}_g \,|\, SM = \tilde{\omega}_g.$$

We can extract from diagram (5.1) the following commuting diagram:

$$\begin{array}{ccc} SL(2,\mathbf{C}) & \overset{\tilde{j}}{\hookrightarrow} & \widetilde{GL^+(4,\mathbf{R})} \\ \Lambda \downarrow & & \tilde{\Lambda} \downarrow \\ SO^o(1,3) & \overset{j}{\hookrightarrow} & GL^+(4,\mathbf{R}) \end{array} \tag{5.5}$$

Since the Lie algebra of a Lie group G can be identified with T_eG, the corresponding diagram of Lie algebras gives us that $\tilde{j}_* \circ \Lambda_*^{-1} = \tilde{\Lambda}_*^{-1} \circ j_*$. Now let $X \in T(SM)$. Then

$$\begin{aligned} \tilde{\imath}^*\tilde{\bar{\omega}}_g(X) &= \tilde{\Lambda}_*^{-1} \circ \tilde{\lambda}^*\bar{\omega}_g(\tilde{\imath}_*X) \\ &= \tilde{\Lambda}_*^{-1} \circ \bar{\omega}_g(\tilde{\lambda}_*\tilde{\imath}_*X) \\ &= \tilde{\Lambda}_*^{-1} \circ \bar{\omega}_g(i_*\lambda_*X) \\ &= \tilde{\Lambda}_*^{-1} \circ \lambda^* i^* \bar{\omega}_g(X) \\ &= \tilde{\Lambda}_*^{-1} \circ \lambda^*(j_* \circ \omega_g)(X) \\ &= \tilde{\Lambda}_*^{-1} \circ j_* \circ \lambda^*\omega_g(X) \\ &= \tilde{j}_* \circ \Lambda_*^{-1} \circ \lambda^*\omega_g(X) \\ &= \tilde{j}_* \circ \tilde{\omega}_g(X). \end{aligned} \tag{5.6}$$

Thus the metric connection geometry on $\widetilde{L^oM}$ lifts naturally from the base frame bundle L^oM and is shown to be an an extension of the natural torsionless spin connection on SM.

6 Hamiltonian vector fields relative to a metric function on $\widetilde{L^oM}$

Now that we have recalled [2] the generalization of a spin structure over the frame bundle $\widetilde{L^oM}$, we can consider finding systems of Hamiltonian vector fields for tensorial metric functions induced on $\widetilde{L^oM}$. It is proven [8] that the Hamiltonian vector fields $X^i_{\tilde{g}}$ are tangent to OM as a subbundle of LM. We wish to investigate the analogue on $\widetilde{L^oM}$.

Note that we may extend $\tilde{\theta}$ to $\widetilde{L^oM}$, i.e., define $\tilde{\theta} := \tilde{\lambda}^*\theta$ on $\widetilde{L^oM}$. The $\mathbf{R}^4$-valued one-form $\tilde{\theta}$ is tensorial relative to the representation $\widetilde{GL^+(4,\mathbf{R})} \to GL(4,\mathbf{R})$ given by $a \cdot v = \tilde{\Lambda}(a)(v)$ for $a \in \widetilde{GL^+(4,\mathbf{R})}$ and $v \in \mathbf{R}^4$. This implies that $d\tilde{\theta}$ is a nondegenerate closed tensorial two-form on $\widetilde{L^oM}$ and thus provides $\widetilde{L^oM}$ with a 4-symplectic structure.

We now seek $\mathbf{R}^4$-valued vector fields $X^i_{\tilde{g}}$ on $\widetilde{L^oM}$ that are solutions to the system

$$d\tilde{g}^{ij} = -2X^{(i}_{\tilde{g}} \lrcorner \; d\tilde{\theta}^{j)}. \tag{6.1}$$

It is convenient to introduce local coordinates on $\widetilde{L^oM}$. In order to define a chart of $\widetilde{L^oM}$ in a neighborhood of $\tilde{u} \in \widetilde{L^oM}$, first choose a local chart (x^i, U) at $p := \tilde{\pi}(\tilde{u}) \in U$, where $\tilde{\pi} = \pi \circ \tilde{\lambda}$ is the canonical projection of $\widetilde{L^oM}$ onto M. Then, using a local trivialization, $\psi : \tilde{\pi}^{-1}(U) \cong U \times \widetilde{GL^+(4,\mathbf{R})}$ and the universal covering property of $\widetilde{GL^+(4,\mathbf{R})}$, one has that for $\psi(\tilde{u}) = (p, g)$ there is an open neighborhood $V \subset \widetilde{GL^+(4,\mathbf{R})}$ containing g such that $\mathrm{id} \times \tilde{\Lambda} \,\big|\, U \times V$ is a diffeomorphism onto $U \times \tilde{\Lambda}(V)$ that respects the right action. We may now define coordinates $\{\tilde{x}^i, \tilde{\pi}^i_j\}$ from an open neighborhood of $\tilde{u}$ in $\tilde{\pi}^{-1}(U)$ to $x(U) \times \tilde{\Lambda}(V)$,

$$\left\{ \begin{array}{lcl} \tilde{x}^i & = & x^i \circ \tilde{\lambda} \\ \tilde{\pi}^i_j & = & \pi^i_j \circ \tilde{\lambda}. \end{array} \right. \tag{6.2}$$

Since we have defined the symplectic structure and tensorial metric function as pullbacks via $\tilde{\lambda}$ of the respective objects on L^oM, we can find local expressions on $U \times V$ for the Hamiltonian vector fields using the same calculations as on L^oM. Invoking the analogous zero-torsion property of the lifted Levi-Civita connection,

$$X^i_{\tilde{g}} \lrcorner \; X^j_{\tilde{g}} \lrcorner \; d\tilde{\theta}^k = 0 \quad \forall\, i, j, k = 0, \ldots, 3\ . \tag{6.3}$$

we obtain

$$X^i_{\tilde{g}} = \tilde{g}^{ab}\tilde{\pi}^i_a \frac{\partial}{\partial \tilde{x}^b} + \tilde{\Gamma}^b_{jc}\tilde{g}^{ac}\tilde{\pi}^i_a\tilde{\pi}^k_b \frac{\partial}{\partial \tilde{\pi}^k_j} \tag{6.4}$$

where $\tilde{\Gamma}^b_{jc}$ are the Christoffel symbols of the lifted Levi-Civita connection. Thus we obtain the parallel transport of linear frames and coframes along timelike geodesics of Γ,

$$\frac{d^2\tilde{x}^i}{dt^2} + \tilde{\Gamma}^i_{jk}\frac{d\tilde{x}^j}{dt}\frac{d\tilde{x}^k}{dt} = 0\ , \tag{6.5}$$

and

$$\frac{D\tilde{\pi}_j^\alpha}{Dt} = \frac{d\tilde{\pi}_j^\alpha}{dt} - \tilde{\Gamma}^k_{ij}\frac{d\tilde{x}^i}{dt}\tilde{\pi}_k^\alpha = 0 \ , \quad \alpha = 1,2,3 \ . \tag{6.6}$$

when restricted to the open neighborhood $U \times V$ of $\widetilde{L^oM}$. This is in exact analogy to that of the n-symplectic theory on LM [8]. We can proceed to verify that the horizontal distribution and thus the set of Hamiltonian vector fields is tangent to SM. So SM is an "integral" of the set of Hamiltonian vector fields and is analogous to constant-energy surfaces in standard symplectic geometry.

7 Initiation of prequantization procedure

Now that we have in hand the geometry of spin bundles we turn to a quantum mechanical application. We want to recast the fundamentals of the Kostant–Souriau theory of geometric quantization [12], taking for the 4-symplectic manifold the prolongation $\widetilde{L^oM}$ of the bundle of linear frames of spacetime M with the 4-symplectic two-form $d\tilde{\theta}$. We restrict attention to the essentials of the initial pre-quantization procedure.

In the naive prequantization program one assigns to each observable $f : T^*M \to \mathbf{R}$ a Hermitian operator

$$f \longrightarrow \mathcal{P}_f = -i\hbar X_f \ . \tag{7.1}$$

The linear operator $\mathcal{P}_f$ acts on the set of square integrable functions $\psi : T^*M \to \mathbf{C}$, square integrability being defined with respect to the natural Liouville volume element on T^*M. Although the assignment (7.1) is not the correct assignment in the full geometric quantization theory, it will suffice for our purposes here.

We consider a spacetime $(M, \vec{g})$ which admits a spin structure. Let $X^i_{\tilde{g}}$ be the set of Hamiltonian vector fields on SM, where SM is viewed as a subbundle of $\widetilde{L^oM}$. Consider the prequantization operator assignments that one can make for the metric tensor Hamiltonian observable on $\widetilde{L^oM}$. The natural analogue of (7.1) is

$$\tilde{g} \to \mathcal{P}_{\tilde{g}} = -i\hbar \hat{X}_{\tilde{g}} = -i\hbar X^i_{\tilde{g}} r_i \ , \tag{7.2}$$

with the $X^i_{\tilde{g}}$ given in (6.4).

We consider the spinor representation of the operator $\mathcal{P}_{\tilde{g}}$ on $L_2(SM, \mathbf{C}^4)$, which we define by

$$\tilde{g} \longrightarrow \mathcal{P}_{\tilde{g}} \longrightarrow -\gamma_i \mathcal{P}^i_{\tilde{g}} = i\hbar\gamma_i X^i_{\tilde{g}} = i\hbar\gamma^i B_i \ , \tag{7.3}$$

where the four γ_i are appropriate Dirac matrices. In writing equation (7.3) we note that the $B_i = \tilde{g}_{ij} X^i_{\tilde{g}}$ are the horizontal vector fields defined by the Levi-Civita connection on $\widetilde{L^oM}$. It follows that $-\gamma_i \mathcal{P}^i_{\tilde{g}}$ in equation (7.3) is the Dirac operator on SM [1].

Let $\Psi \in L_2(SM, \mathbf{C}^4)$ be a Dirac 4-spinor transforming under $SL(2, \mathbf{C})$ transformations on the spin bundle as

$$\Psi(u \cdot a) = \rho(a^{-1}) \cdot \Psi(u) \ , \ \forall\ u \in SM \ , \ \forall\ a \in SL(2, \mathbf{C}) \ , \tag{7.4}$$

where ρ denotes the 4-spinor representation of $SL(2,\mathbf{C})$. Then it follows that

$$\gamma_i \mathcal{P}^i_{\tilde{g}}(\Psi)(u \cdot a) = \rho(a^{-1}) \cdot \gamma_i \mathcal{P}^i_{\tilde{g}}(\Psi)(u) \tag{7.5}$$

Thus the eigenvalue equation

$$-\gamma_i \mathcal{P}^i_{\tilde{g}}(\Psi) = \mu\Psi \tag{7.6}$$

for the prequantization operator $\gamma_i \mathcal{P}^i_{\tilde{g}}$ is tensorial on SM and is in fact just the Dirac equation

$$i\hbar\gamma^i B_i(\Psi) = \mu\Psi \quad . \tag{7.7}$$

8 Conclusions

This work incorporating geometry into the physics of spinning particles is significant in view of the central role played by the Dirac equation in all relativistic theories involving the half-integer spin properties of fermions. Moreover, our work addresses in a rather direct way one of the most fundamental features of the quantum theory, namely the unavoidable interaction of observer and object. Indeed, since the orthonormal frame bundle OM may be considered the "bundle of Lorentzian observers," the bundle OM together with the 4-symplectic structure might be called the "phase space of relativistic observers". This seems an ideal setting to consider questions relating to the quantum mechanical phenomenon of the interaction of observer and object. Indeed, the spin bundle SM covering OM has structure group the universal cover of the Lorentz group [1] and it also has a previously unrecognized 4-symplectic structure that, following a natural geometrical construction, leads easily to the correct relativistic equation for the fermions. Because our procedure parallels those of standard geometric quantization on symplectic manifolds, we are led to investigate the construction of the Hilbert space of prequantization, the frame bundle analogue of the line bundle, polarizations on this frame bundle, and initiation of the quantization procedure. We hope to return to these questions in later publications.

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