Covariant Field Theory

on

Frame Bundles of Fibered Manifolds†

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Abstract

We show that covariant field theory for sections of $\pi : E \to M$ lifts in a natural way to the bundle of vertically adapted linear frames $L_\pi E$. Our analysis is based on the fact that $L_\pi E$ is a principal fiber bundle over the bundle of 1-jets $J^1\pi$. On $L_\pi E$ the canonical soldering 1-forms play the role of the contact structure of $J^1\pi$. A lifted Lagrangian $\mathcal{L}$:$L_\pi E \to \mathbb{R}$ is used to construct modified soldering 1-forms, which we refer to as the Cartan-Hamilton-Poincaré 1-forms. These 1-forms on $L_\pi E$ pass to the quotient to define the standard Cartan-Hamilton-Poincaré m-form on $J^1\pi$. We derive generalized Hamilton-Jacobi and Hamilton equations on $L_\pi E$, and show that the Hamilton-Jacobi and canonical equations of Carathéodory-Rund and de Donder-Weyl are obtained as special cases. The manifold $L_\pi E$ emerges as a natural arena for a unified theory that contains, in addition to the sector for sections of π, dynamical sectors for a geometry for M and a geometry for the fibers of E.

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1 Introduction

The Cartan-Hamilton-Poincaré (CHP) m-form is the central object in covariant Lagrangian field theory. The ingredients which go into the construction of this m-form are:

1. A Lagrangian $\mathrm{L} : J^1\pi \to \mathbb{R}$, on the bundle of 1-jets of sections of $\pi : E \to M$, where E is the configuration manifold of the theory,

2. A volume on the m-dimensional parameter space M,

3. The contact structure of $J^1\pi$.

It is the contact structure [17] in this mixture of ingredients that provides the geometrical foundation of the theory. In this paper we give a new geometrical formulation of the covariant field theory on $J^1\pi$ by lifting it to the bundle of vertically adapted linear frames $L_\pi E$ of E. We will show that the full depth of Lagrangian and Hamiltonian field theory on $J^1\pi$ has a useful geometrical representation on the bundle $L_\pi E$. In this representation the role of the contact structure of $J^1\pi$ is taken over by the canonical vector-valued soldering 1-form on $L_\pi E$. Introduction of a Lagrangian leads to the definition of a modified soldering form, and this vector-valued 1-form plays the role of the CHP-m-form. These structures pass to a certain quotient of $L_\pi E$ to give the standard structures on $J^1\pi$. The advantaged gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L_\pi E$, namely n-symplectic geometry, to further develop covariant field theory.

If E is an arbitrary n-dimensional manifold, then the bundle of linear frames $\lambda : LE \to E$ supports a canonically defined $\mathbb{R}^n$-valued 1-form, the "soldering" 1-form. n-symplectic geometry on LE is the generalized symplectic geometry that emerges upon taking the soldering 1-form θ as the generalized symplectic potential. This geometry, including the notions of n-symplectic observables, the corresponding generalized Hamiltonian vector-valued vector fields, and generalized Poisson and graded Poisson brackets, has been developed in a series of papers [5, 6, 12, 13, 14, 15]. A sketch of the basic structure of the theory can be found in section 2, but let us point out here that in [14] it was shown that the fundamentals of the canonical symplectic geometry on the cotangent bundle T^*E can be constructed entirely in terms of the n-symplectic geometry on LE.

When E has extra structure, in particular when $\pi : E \to M$ is a fiber bundle as it is in Lagrangian field theory, then the n-symplectic geometry likewise inherits extra structure on LE. In particular, the fiber structure of $\pi : E \to M$ leads to a reduction of LE to the subbundle of vertically adapted linear frames $L_\pi E$, with a corresponding reduction in the generality of n-symplectic observables. The structure group $\mathrm{G_v}$ of $L_\pi E$ is the subgroup of $\mathrm{GL}(n)$ that is block lower triangular, corresponding to the convention that the last $k = n-m$ vectors in each linear frame are required to be vertical. Following the model construction given in [14] Lawson showed [11, 6] that the multisymplectic geometry [7] on the affine cojet bundle $J^{1*}\pi$ can also be derived directly from the n-symplectic geometry on $L_\pi E$.

Turning our attention in this paper to the covariant field theory on $J^1\pi$, we will show that the geometrical foundations of the theory, namely the contact structure on $J^1\pi$, follows directly from the n-symplectic structure on $L_\pi E$, while the CHP-form follows from a modified soldering form. The central idea on which the analysis is based is the following theorem.

Theorem 1.1 *Let $\pi : E \to M$ be an $m+k$ dimensional fiber bundle over the m-dimensional manifold M. The vertically adapted frame bundle $L_\pi E$ is a principal $H = \mathrm{GL}(m) \times \mathrm{GL}(k)$ bundle over $J^1\pi$. In particular, $J^1\pi \cong L_\pi E/H$.*

As a consequence of this theorem, which we prove in section 3, the canonical soldering forms on $L_\pi E$ pass to the quotient to define the contact structure of $J^1\pi$ (see section 5).

Furthermore, this theorem leads to a decomposition of $L_\pi E$. Once a Lagrangian is introduced, this decomposition will lead us to larger theory that is a type of Kaluza-Klein theory that includes a dynamical sector for a geometry of the parameter space ("spacetime") M, a dynamical sector for a geometry of the fibers of E, in addition to the original sector for the sections of E. A simple picture of this development can be sketched out as follows.

Let (x^i) be local coordinates on M and let (y^A) be fiber coordinates on E, so that $(z^\alpha) = (x^i, y^A)$ are adapted local coordinates on E. With respect to such coordinates a general vertically adapted linear frame at a point in E will be of the form

$$\begin{array}{l} (e_i, e_A) = (v_i^j \frac{\partial}{\partial x^j} + v_i^B \frac{\partial}{\partial y^B}, v_A^B \frac{\partial}{\partial y^B}) \\ i, j = 1, \ldots, m, \quad A, B = m+1, \ldots, m+k \end{array}$$

The first m vectors (e_i) are non-vertical while the last k vectors (e_A) are vertical with respect to π. The matrices (v_i^j) and (v_A^B) are necessarily non-singular, while the matrix $(v_i^B) \in \mathbb{R}^{k\times m}$

is arbitrary. Hence we may take the collection $(x^i, y^A, v^j_i, v^B_i, v^B_A)$ as local coordinates on $L_\pi E$. We can represent an arbitrary adapted linear frame in terms of these local coordinates as the $(m+k)\times(m+k)$ matrix

$$\begin{pmatrix} v^j_i & 0 \\ v^B_i & v^B_A \end{pmatrix}$$

Using the notation $\pi^i_j = (v^i_j)^{-1}$ and $\pi^B_A = (v^B_A)^{-1}$, this matrix can be decomposed as follows:

$$\begin{pmatrix} v^j_i & 0 \\ v^B_i & v^B_A \end{pmatrix} = \begin{pmatrix} \delta^j_k & 0 \\ \pi^a_k v^B_a & \delta^B_C \end{pmatrix} \begin{pmatrix} v^k_i & 0 \\ 0 & v^C_A \end{pmatrix} \tag{1.1}$$

The first factor is H invariant and defines a natural projection to $J^1\pi$. We thus obtain the decomposition

$$L_\pi E = J^1\pi \times_E (LVE \times_M LM)$$

where LVE denotes the bundle of vertical frames of E.

These results suggest that it may be useful to lift the covariant Lagrangian field theory on $J^1\pi$ to $L_\pi E$. In particular on $L_\pi E$ we have available the n-symplectic geometry to use in studying the structure of field theories. We show in section 4 that for a lifted Lagrangian $\mathcal{L} = \rho^*(\mathrm{L})$, the n-symplectic Hamiltonian vector fields defined by vertical vector fields on E may be thought of as *variational vector fields.* If X is such a vector field then $X(\mathcal{L})$ gives the Euler-Lagrange operator to within a total divergence.

In section 6 we turn to the problem of constructing, on $L_\pi E$, a lifted version of the CHP m-form. We show that in fact one can use a lifted Lagrangian to define an $\mathbb{R}^n$-valued CHP-form using the canonical $\mathbb{R}^n$-valued soldering form θ. The key to the construction is to use the fundamental vertical vector fields on $L_\pi E$ together with the Lagrangian to give a global, invariant definition of the covariant momentum, which is essentially a frame bundle version of the Legendre transformation of classical theory. The result is that the $\mathbb{R}^n$-valued CHP-form is a *modified, or non-canonical soldering form* $\theta_{\mathcal{L}}$. This new vector-valued CHP-form $\theta_{\mathcal{L}}$ passes to the quotient to define the standard CHP-m-form on $J^1\pi$.

As an application of the general formalism we derive in section 7 a generalized Hamilton-Jacobi differential equation and generalized Hamilton equations. Under appropriate assumptions these equations reproduce the Hamilton-Jacobi equations and Hamilton equations of the de Donder-Weyl [4, 16] and Carathéodory-Rund [2, 16] theories.

We recall that there is a certain degree of arbitrariness in Rund's [16] canonical formalism for Carathéodory's theory. We find that by identifying the canonical variables introduced here with the canonical variables in Rund's formalism, the undetermined features of the Carathéodory-Rund theory can be given a natural interpretation on $L_\pi E$, namely as the variables defining linear frames for M. Looking again at the decomposition (1.1) we see now that the entries in the right-hand-factor represent a linear frame for M (the (v_i^j) factor) together with a linear frame for the fibers of E (the (v_A^B) factor). Thus by dropping the condition that the Lagrangian $\mathcal{L} : L_\pi E \to \mathbb{R}$ be a lifted Lagrangian, one arrives at a theory where the solutions of the Euler-Lagrange field equations would determine not only a section of π, but also a linear frame field for M together with a linear frame field for the fibers of E. We present a model Lagrangian in section 9 that describes a Kaluza-Klein type theory, formulated in a natural way on $L_\pi E$. Section 10 contains concluding remarks together with plans for applications and extensions of the results presented in this paper.

2 The Vertically Adapted Linear Frame Bundle $L_\pi E$

Let $\pi : E \to M$ be a fiber bundle where M is m-dimensional and E is $n = m+k$-dimensional. Lower case latin indices are assumed to range over $1 \ldots m$, upper case latin indices over $m+1 \ldots m+k$, and greek indices over $1 \ldots m+k$. This convention will be used throughout the paper.

An adapted frame at $e \in E$ is a frame where the last k basis vectors are vertical. Note that coordinate frames that come from adapted coordinates are adapted frames. The adapted frame bundle of π, denoted $L_\pi E$, consists of all adapted frames for E.

$$L_\pi E = \{(e, \{e_i, e_A\}) : e \in E, \{e_i, e_A\} \text{ is a basis for } T_e E, \text{ and } d_u\pi(e_A) = 0\}$$

The canonical projection, $\lambda : L_\pi E \to E$, is defined by $\lambda(e, \{e_i, e_A\}) = e$.

$L_\pi E$ is a reduced subbundle of LE, the frame bundle of E (Lawson [11]). As such it is a principal fiber bundle over E. Its structure group is $\mathrm{G_v}$, the nonsingular block lower triangular matrices.

$$\mathrm{G_v} = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} : A \in \mathrm{GL}(m), B \in \mathrm{GL}(k), C \in \mathbb{R}^{km} \right\}$$

G_v acts on $L_\pi E$ on the right by

$$(e,\{e_i,e_A\})\cdot\begin{pmatrix} A & 0 \\ C & B\end{pmatrix} = \{(e,\{e_iA^i_j + e_AC^A_j, e_AB^A_B\})$$

2.1 Coordinates

If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$, then one may induce several different coordinates on $\lambda^{-1}(U)$. First consider the *coframe* or *n-symplectic momentum* coordinates $(x^i, y^A, \pi^i_j, \pi^A_j, \pi^A_B)$ on $\lambda^{-1}(U)$ defined by

$$x^i(e,\{e_i,e_A\}) = x^i(e) \qquad \pi^i_j(e,\{e_i,e_A\}) = e^i(\frac{\partial}{\partial x^j}) \qquad \pi^A_B(e,\{e_i,e_A\}) = e^A(\frac{\partial}{\partial y^B})$$

$$y^A(e,\{e_i,e_A\}) = y^A(e) \qquad \pi^A_j(e,\{e_i,e_A\}) = e^A(\frac{\partial}{\partial x^j})$$

Here (e^i, e^A) is the dual frame to (e_i, e_A). We have as is customary retained the same symbols for the induced horizontal coordinates.

Secondly consider the *frame* or *n-symplectic velocity* coordinates $(x^i, y^A, v^i_j, v^A_j, v^A_B)$ on $\lambda^{-1}(U)$ defined by

$$x^i(e,\{e_i,e_A\}) = x^i(e) \qquad v^i_j(e,\{e_i,e_A\}) = e_j(x^i) \qquad v^A_B(e,\{e_i,e_A\}) = e_B(y^A)$$

$$y^A(e,\{e_i,e_A\}) = y^A(e) \qquad v^A_j(e,\{e_i,e_A\}) = e_j(y^A)$$

The v coordinates, viewed together as a block triangular matrix, form the inverse of the π coordinates above. The blocks have the following relations:

$$v^i_j\pi^j_k = \delta^i_k \qquad v^A_j\pi^j_k + v^A_B\pi^B_k = 0 \qquad v^A_B\pi^B_C = \delta^A_C$$

Lastly consider the following coordinates which are constructed from the previous two. Define $(x^i, y^A, u^i_j, u^A_j, u^A_B)$ on $\lambda^{-1}(U)$ by

$$x^i(e,\{e_i,e_A\}) = x^i(e) \qquad u^i_j = \pi^i_j \qquad u^A_j = v^A_i\pi^i_j = -v^A_B\pi^B_j$$

$$y^A(e,\{e_i,e_A\}) = y^A(e) \qquad u^A_B = \pi^A_B$$

It will turn out that the u^A_j coordinates are pull-ups of the standard jet coordinates on $J^1\pi$. As such, we will refer to these coordinates as *Lagrangian* coordinates.

Later in the paper we will need the following formulas for the fundamental vertical vector fields $E^{*\alpha}_{\beta}$ on $L_\pi E$ in Lagrangian coordinates.

$$E^{*i}_{j} = -u^i_k \frac{\partial}{\partial u^j_k} \qquad E^{*A}_{B} = -u^A_C \frac{\partial}{\partial u^B_C} \qquad E^{*i}_{A} = u^i_k v^B_A \frac{\partial}{\partial u^B_k} \tag{2.2}$$

2.2 n-symplectic structure

n-symplectic geometry arises naturally on the frame bundle LE of any n-dimensional manifold E. LE supports a canonically defined $\mathbb{R}^n$-valued 1-form θ, the soldering 1-form, and n-symplectic geometry is the geometry on LE when one takes $d\theta$ as a vector-valued generalized symplectic form. We present here a sketch of the structure of the theory and refer the reader to the literature [5, 6, 12, 13, 14, 15] for more details. See also the works of de León, Salgado et al. [3] and Awane [1].

The intrinsic definition of the soldering 1-form θ parallels the definition of the canonical form on T^*M.

$$\theta_u(X) = e^\alpha(d_u\bar{\lambda}(X))r_\alpha = \theta^\alpha_u(X)r_\alpha \tag{2.3}$$

Here $u = (e, \{e_\alpha\}) \in LE$, $\bar{\lambda} : LE \to E$ is the canonical projection, and $\{r_\alpha\}$ is the standard basis for $\mathbb{R}^n$. In canonical coordinates,

$$\theta^\alpha = \pi^\alpha_\beta dx^\beta$$

The above formula parallels the local coordinate formula $\vartheta = p_i dq^i$ for the canonical 1-form on T^*M.

Because the soldering 1-form θ is vector-valued, the natural structure equation for n-symplectic geometry takes the generalized form

$$d\hat{f}^{\alpha_1\alpha_2\cdots\alpha_p} = -p! X_{\hat{f}}{}^{\alpha_1\alpha_2\cdots\alpha_{p-1}} \,\lrcorner\, d\theta^{\alpha_p} \tag{2.4}$$

Here $\hat{f} = (\hat{f}^{\alpha_1\alpha_2\cdots\alpha_p}) : LE \to \otimes^p\mathbb{R}^n$ is a vector-valued function on LE and $X_{\hat{f}} = (X_{\hat{f}}{}^{\alpha_1\alpha_2\cdots\alpha_{p-1}})$ is the corresponding set of Hamiltonian vector fields. (Each superscript α_k, $k = 1, 2, \ldots, p$, runs from 1 to n). Moreover, since the soldering form is equivariant under the free right action of the structure group $GL(n, \mathbb{R})$ on LE, the class of functions that can satisfy (2.4) is restricted. They divide naturally into vector-valued functions that map to either the symmetric tensor spaces $(\otimes_s)^p\mathbb{R}^n$ or the anti-symmetric tensor spaces $(\otimes_a)^p\mathbb{R}^n$, where $\otimes_s$ and $\otimes_a$ denote the symmetric and anti-symmetric tensor products, respectively. There is a *naturally defined Poisson bracket* for both sets of observables, and the complete set of symmetric observables is a *Poisson algebra* with respect to the bracket, while the set of anti-symmetric observables is a *graded Poisson algebra* with respect to the bracket. These brackets, when restricted to the subsets of tensorial observables, are the frame bundle versions of the Schouten-Nijenhuis brackets [15]. On $L_\pi E$ the allowable tensorial observables [11] correspond to contravariant tensor fields that are projectible to E.

As a reduced subbundle of LE, $L_\pi E$ has the n-symplectic geometry obtained by restricting the soldering form. Since this soldering form is $\mathbb{R}^{m+k}$-valued, we will denote it (θ^i, θ^A). Let $u = (e, \{e_i, e_A\})$ be a point in $L_\pi E$. If $\lambda : L_\pi E \to E$ is the canonical projection and $X \in T_u L_\pi E$, then θ defined as in (2.3) above splits naturally into the two terms

$$\theta_u(X) = \theta^i(X)r_i + \theta^A(X)r_A$$

where (e^i, e^A) is the dual frame and (r_i, r_A) is the standard basis for $\mathbb{R}^{m+k}$. In local momentum coordinates,

$$\theta^i = \pi^i_j dx^j \qquad \theta^A = \pi^A_j dx^j + \pi^A_B dy^B$$

3 The Relationship between $L_\pi E$ and $J^1\pi$

We will demonstrate three useful facts relating $L_\pi E$ and $J^1\pi$.

1. $J^1\pi$ is an associated bundle to $L_\pi E$ [11].
2. $L_\pi E$ is a principal fiber bundle over $J^1\pi$.
3. $L_\pi E$ is a pull-back bundle over $J^1\pi$ [11].

3.1 A special case

Consider the case where π is a trivial bundle. Let $M = \mathbb{R}^m$ and $E = \mathbb{R}^m \times F$ with F a k-dimensional manifold. Let $\pi : \mathbb{R}^m \times F \to \mathbb{R}^m$ be the standard projection. It is known that for this bundle each 1-jet corresponds to an m-tuple of tangent vectors to F.

$$J^1\pi \cong \mathbb{R}^m \times (TF \oplus \cdots \oplus TF)$$

It is clear that such a bundle is associated to $L_\pi E$.

Let us examine $L_\pi E$ in this case. We will make use of the other projection mapping $\bar{\pi} : \mathbb{R}^m \times F \to F$. For each frame $(u, \{e_i, e_A\})$ in $L_\pi E$, we decompose each vector into

$$e_i = (v_i, w_i) \quad e_A = (v_A, w_A)$$

where $v_i = d_u\pi(e_i)$, $w_i = d_u\bar{\pi}(e_i)$, $v_A = d_u\pi(e_A)$, and $w_A = d_u\bar{\pi}(e_A)$. Note that $v_A = 0$ by the definition of $L_\pi E$, so we have

$$e_i = (v_i, w_i) \quad e_A = (0, w_A)$$

The k vectors $\{w_A\}$ form a basis for $\mathrm{T}_{\bar{\pi}(u)}\,\mathrm{F}$, and the m vectors $\{v_i\}$ form a basis for $\mathrm{T}_{\pi(u)}\,\mathbb{R}^m$. The m vectors $\{w_i\}$ are simply an m-tuple of vectors in $\mathrm{T}_{\bar{\pi}(u)}\,\mathrm{F}$.

Decomposing all of $L_\pi E$ in this way, we obtain

$$L_\pi E \cong J^1\pi \times_E (\mathrm{L}\,\mathbb{R}^m \times \mathrm{LF})$$

This is a bundle isomorphism over $E = \mathbb{R}^m \times F$. From this decomposition, it is clear that $L_\pi E$ is a pull-back bundle over $J^1\pi$. Furthermore, the fiber is the Lie group $\mathrm{GL}(m) \times \mathrm{GL}(k)$.

3.2 The general case

Consider an arbitrary fiber bundle $\pi : E \to M$. In this more general setting, a 1-jet is no longer simply an m-tuple of tangent vectors. There are three major ways of describing 1-jets, each with its own charm:

1. Equivalence classes of sections of π.

2. Linear right-inverses to $d_u\pi$.

3. Non-vertical m-dimensional subspaces of T_uE.

One quick way to define the projection from $L_\pi E$ to $J^1\pi$ is to map each adapted frame to the span of its non-vertical elements.

$$(u, \{e_i, e_A\}) \mapsto (u, \mathrm{span}\{e_i\})$$

However, we will benefit from starting with $J^1\pi$ as an associated bundle.

As stated earlier, the structure group of $L_\pi E$ is $\mathrm{G_v}$, the nonsingular block lower triangular matrices. This group $\mathrm{G_v}$ can be decomposed [11] into the product of two of its subgroups, H and J, where

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{GL}(m), B \in \mathrm{GL}(k) \right\}$$

and

$$J = \left\{ \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} : C \in \mathbb{R}^{km} \right\}$$

Note that J is Lie group isomorphic to the additive group $\mathbb{R}^{km}$.

We will show that $J^1\pi$ is a bundle associated to $L_\pi E$ with fiber $\mathrm{G_v}/H$. Although H is a closed Lie subgroup of $\mathrm{G_v}$ it is not normal. As such $\mathrm{G_v}/H$ does not have a natural group structure; it is manifold with a left $\mathrm{G_v}$-action. For each coset $gH \in \mathrm{G_v}/H$, we select the unique representative in J.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}$$

By choosing these representatives, we identify $\mathrm{G_v}/H$ with J and hence $\mathbb{R}^{km}$. These identifications are diffeomorphisms.

Consider how the left $\mathrm{G_v}$-action looks for our selected representatives.

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \begin{pmatrix} I & 0 \\ \xi & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ C + B\xi & B \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ CA^{-1} + B\xi A^{-1} & I \end{pmatrix}$$

So the $\mathrm{G_v}$-action appears *affine* when $\mathrm{G_v}/H$ is identified with $\mathbb{R}^{km}$. Therefore it is prudent to use this identification to define an affine structure on $\mathrm{G_v}/H$ modelled on $\mathbb{R}^{km}$. This $\mathrm{G_v}$-invariant structure will pass to the fibers of the associated bundle, making it an affine bundle.

Theorem 3.1 $\quad L_\pi E \times_{\mathrm{G_v}} (\mathrm{G_v}/H) \cong J^1\pi$

Proof: The isomorphism maps each equivalence class $[(e, \{e_i, e_A\}, \xi)]$ to the linear map $\phi : T_{\pi(e)}M \to T_eE$ defined by $\phi(\hat{e}_i) = e_i + \xi_i^A e_A$, where we use the basis $\hat{e}_i = d_e\pi(e_i)$. ■

Corollary 3.2 *$L_\pi E$ is a principal fiber bundle over $J^1\pi$ with fiber H.*

Proof: This fact follows directly from proposition 5.5 in reference [10]. ■

We will denote the projection from $L_\pi E$ to $J^1\pi$ by ρ. It is given by

$$\rho(e, \{e_i, e_A\}) = (e, \tau) \quad \text{where} \;\; \tau(\hat{e}_i) = e_i$$

We now show that the u_j^A-coordinates defined earlier are the pull-ups of the jet coordinates. If (x^i, y^A) are adapted coordinates on an open set $U \subseteq E$ and $u = (e, \{e_i, e_A\}) \in \lambda^{-1}(U)$ then

$$\begin{aligned} y_i^A \circ \rho(u) = y_i^A(e, \tau) &= d_e y^A \circ \tau\Big(\frac{\partial}{\partial x^i}\Big|_{\pi(e)}\Big) = d_e y^A\Big(e_j \hat{e}^j\Big(\frac{\partial}{\partial x^i}\Big|_{\pi(e)}\Big)\Big) \\ &= d_e y^A(e_j) e^j\Big(\frac{\partial}{\partial x^i}\Big|_{e}\Big) = v_j^A(u)\pi_i^j(u) = u_j^A(u) \end{aligned}$$

What remains to be shown is that $L_\pi E$ is a pull-back bundle over $J^1\pi$. To see this, we will decompose each adapted frame in a manner similar to the trivial case covered earlier. We can split each adapted frame (u, e_i, e_A) into three pieces:

1. A point in LM, $(\pi(u), \tilde{e}_i)$, where $\tilde{e}_i = d_u\pi(e_i)$
2. A point in LVE, (u, e_A), where LVE is the bundle of vertical frames over E
3. A point in $J^1\pi$, (u, ϕ), where $\phi : T_{\pi(u)}M \to T_uE$ is defined by $\phi(\tilde{e}_i) = e_i$

Theorem 3.3 $\quad L_\pi E \cong J^1\pi \times_E (LVE \times_M LM)$

Proof: The isomorphism is given by $(u, e_i, e_A) \mapsto \big((u, \phi), (u, e_A), (\pi(u), \tilde{e}_i)\big)$. The inverse map is quite nice: $\big((u, \phi), (u, f_A), (p, f_i)\big) \mapsto (u, \phi(f_i), f_A)$ ■

4 Prolongations of Vector Fields to $L_\pi E$

Definition 4.1 *A Lagrangian on $L_\pi E$ is a function $\mathcal{L} : L_\pi E \to \mathbb{R}$. A Lagrangian on $L_\pi E$ is* **lifted** *if it satisfies the auxiliary conditions*

$$E^{*i}_j(\mathcal{L}) = 0 \qquad E^{*A}_B(\mathcal{L}) = 0$$

Remark Using (2.2) one can show that these conditions imply that $\mathcal{L}$ is constant on the fibers of $\rho : L_\pi E \to J^1\pi$, and thus is the pull up of a function on $J^1\pi$. We will assume that our Lagrangians are **lifted** until section 9 where we will drop this assumption in order to study the extra $\mathrm{GL}(m) \times \mathrm{GL}(k)$ degrees of freedom in this bundle structure.

In order to see the role played by the canonical n-symplectic structure on $L_\pi E$ in Lagrangian field theory, we consider a variation of a local section $\phi : M \to E$. The variation of ϕ can be defined by a vector field f on E that projects to the zero vector field on M, so that in adapted local coordinates f has the form $f = f^A \partial_A$. The associated tensorial function $\hat{f} : L_\pi E \to \mathbb{R}^{m+k}$ is given in local coordinates on $L_\pi E$ by $\hat{f} = \hat{f}^\alpha \hat{r}_\alpha$, where

$$(\hat{f}^\alpha) = (\hat{f}^i, \hat{f}^A) = (0, f^B \pi^A_B)$$

The n-symplectic Hamiltonian vector field $X_{\hat{f}}$ determined by $\hat{f}$ is the unique solution of equation (2.4) with $p = 1$. Thus $X_{\hat{f}}$ is defined by

$$d\hat{f}^\alpha = -X_{\hat{f}} \rfloor \, d\theta^\alpha$$

and in local coordinates it has the form [12]

$$X_{\hat{f}} = f^A \partial_A - \frac{\partial f^A}{\partial x^j} \pi^B_A \frac{\partial}{\partial \pi^B_j} - \frac{\partial f^A}{\partial y^C} \pi^B_A \frac{\partial}{\partial \pi^B_C}$$

Transforming to Lagrangian coordinates we find

$$\begin{aligned} X_{\hat{f}} &= \left(f^A \partial_A + \left(\frac{\partial f^A}{\partial x^j} + u^B_j \frac{\partial f^A}{\partial y^B}\right) \frac{\partial}{\partial u^A_j} \right) - \left(\frac{\partial f^A}{\partial y^C} u^B_A\right) \frac{\partial}{\partial u^B_C} \\ &= \left(f^A \partial_A + \frac{df^A}{dx^i} \frac{\partial}{\partial u^A_j} \right) - \left(\frac{\partial f^A}{\partial y^C} u^B_A\right) \frac{\partial}{\partial u^B_C} \end{aligned} \tag{4.5}$$

Lemma 4.2 *Let f be a vertical vector field on E. The projection of the associated Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ to $J^1\pi$ is the prolongation $j(f)$ of f to $J^1\pi$.*

Proof The vector fields $\frac{\partial}{\partial u^B_C}$ are vertical with respect to ρ, and $\rho_*(\frac{\partial}{\partial u^A_j}) = \frac{\partial}{\partial y^A_j}$. ■

This lemma shows that the Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ is a lift of the prolongation of f to $J^1\pi$. That $X_{\hat{f}}$ actually has the properties of the prolongation of f with respect to Lagrangians follows from the following lemma. We let

$$\mathcal{E}_A(\cdot) = \frac{\partial(\cdot)}{\partial y^A} - \frac{d}{dx^i}\left(\frac{\partial(\cdot)}{\partial u^A_i}\right)$$

denote the Euler-Lagrange operator in local coordinates on $L_\pi E$.

Lemma 4.3 *If $X_{\hat{f}}$ is the n-symplectic Hamiltonian vector field on $L_\pi E$ of a vertical vector field f on E, and if $\mathcal{L}$ is a lifted Lagrangian on $L_\pi E$, then*

$$X_{\hat{f}}(\mathcal{L}) = f^A \mathcal{E}_A(\mathcal{L}) + \frac{d}{dx^i}(f^A p^i_A) \tag{4.6}$$

Proof The proof is a straightforward calculation using (4.5). ■

After introducing the CHP 1-forms in the section 6 we will use (4.6) to lift the variational principle to $L_\pi E$.

Remark As mentioned in section 2.2 there are other observables in n-symplectic geometry on $L_\pi E$ beyond those corresponding to vertical vector fields on E. In particular there is the Poisson algebra of all vertical symmetric contravariant tensor fields on E, as well as the graded Poisson algebra of all vertical antisymmetric contravariant tensor fields on E. The associated (equivalence classes of) vector-valued Hamiltonian vector fields on $L_\pi E$ also project to tensor fields on $J^1\pi$. Since these vector-valued Hamiltonian vector fields generalize the natural lift of a vector field from E to $L_\pi E$, their projections to $J^1\pi$ can be taken as the prolongation of the tensor fields on E to $J^1\pi$. These ideas will be elaborated in more detail elsewhere.

5 The Contact Structure

The contact structure on $J^1\pi$ amounts to a natural splitting of the tangent and cotangent spaces to E. For every $(e,\tau) \in J^1\pi$ there is a natural splitting of T_eE and T_eE^* into horizontal and vertical subspaces. This is usually encoded via the linear projections onto the vertical and horizontal. Saunders [17] envisions the contact structure as linear endomorphisms of the pullback vector bundles $J^1\pi \times_E (TE)$ and $J^1\pi \times_E (T^*E)$. These maps can be defined invariantly as follows. For $(e,\tau) \in J^1\pi$, $X \in T_eE$, and $\omega \in T^*_eE$.

$$h(X) = \tau \circ d_e\pi(X) \qquad v(X) = X - h(X)$$
$$h^t(\omega) = \omega \circ \tau \circ d_e\pi \qquad v^t(\omega) = \omega - h^t(\omega)$$

Guillemin and Sternberg [8] prefer to think of the contact structure as TE-valued 1-forms on $J^1\pi$. To achieve this, they compose the h and v above with $d_{(e,\tau)}\pi_{1,0}$, where $\pi_{1,0} : J^1\pi \to E$.

In local coordinates, the contact structure looks like a pair of (1,1) tensor fields on E, except that they depend on jet coordinates.

$$h = dx^k \otimes (\frac{\partial}{\partial x^k} + y^A_k \frac{\partial}{\partial y^A}) \qquad v = (dy^B - y^B_j dx^j) \otimes \frac{\partial}{\partial y^B}$$

Depending on interpretation, the expressions above can be the horizontal and vertical projections for either $J^1\pi \times_E (TE)$ or $J^1\pi \times_E (T^*E)$. They can also be interpreted as TE-valued 1-forms on $J^1\pi$.

5.1 The Contact Structure viewed from $L_\pi E$

The contact structure arises on $J^1\pi$ because each 1-jet (e,τ) allows us to decompose T_eE into a direct sum. Similarly, the soldering form arises on $L_\pi E$ because each adapted frame $u = (e, e_i, e_A)$ allows us to represent T_eE as $\mathbb{R}^{m+k}$. *So the contact structure is analogous to the soldering form.* Recall that

$$\theta_u(X) = e^i(d_u\lambda(X))r_i + e^A(d_u\lambda(X))r_A = \theta^i_u(X)r_i + \theta^A_u(X)r_A$$

and that in local coordinates,

$$\theta^i = \pi^i_j dx^j \qquad \theta^A = \pi^A_j dx^j + \pi^A_B dy^B$$

Consider the following TE-valued one-forms on $L_\pi E$

$$\theta_h(u) = \theta^i(u) \otimes e_i \qquad \theta_v(u) = \theta^A(u) \otimes e_A$$

In local coordinates,

$$\theta_h = \pi^i_k dx^k \otimes v^l_i(\frac{\partial}{\partial x^l} + u^A_l \frac{\partial}{\partial y^A}) = dx^k \otimes (\frac{\partial}{\partial x^k} + u^A_k \frac{\partial}{\partial y^A})$$

$$\theta_v = \pi^A_B(dy^B - u^B_j dx^j) \otimes v^C_A \frac{\partial}{\partial y^C} = (dy^B - u^B_j dx^j) \otimes \frac{\partial}{\partial y^B}$$

These objects are strikingly similar to the contact structure of $J^1\pi$. In fact, they pass to the quotient to give the contact structure on $J^1\pi$. The contact structure is known to appear in "various guises" [17]; the soldering form on $L_\pi E$ is another, perhaps more potent, version.

We also remark that the contact structure falls trivially from the following theorem

Theorem 5.1 *Let $\lambda : P \to E$ be a principal fiber bundle with structure group G, let $H \subseteq G$ be a closed lie subgroup, and let F be a manifold with a left G-action. Then*

$$P \times_H F \cong (P/H) \times_E (P \times_G F)$$

Proof: First note that by Proposition 5.5 in reference [10], $P/H \cong P \times_G (G/H)$ and $\rho : P \to P/H$ is a principal bundle. So $P \times_H F$ is a bundle associated to ρ. This makes sense–if F has a left G-action then it has a left H-action. The isomorphism map is

$$(p, f)H \mapsto (pH, (p, f)G)$$

It is well-defined and a smooth diffeomorphism. ■

Corollary 5.2

$$L_\pi E \times_H \mathbb{R}^{m+k} \cong J^1\pi \times_E TE$$

$$L_\pi E \times_H (\mathbb{R}^{m+k})^* \cong J^1\pi \times_E T^*E$$

The natural splitting of the fibers $\mathbb{R}^{m+k}$ and $(\mathbb{R}^{m+k})^*$ is H-invariant and passes to the quotient to form the contact structure.

6 The Cartan-Hamilton-Poincaré Forms

One associates [8, 7] with a given Lagrangian L on $J^1\pi$ the Cartan-Hamilton-Poincaré (CHP)-m-form θ_{L}, which one may use to reexpress the action integral of the Lagrangian. This form can be defined directly [8] on $J^1\pi$, or it can be defined [7] on $J^1\pi$ as the pull back of the canonical multisymplectic form on $J^{1*}\pi$, the affine dual of $J^1\pi$. Although the CHP-form on $L_\pi E$ can be defined in terms of the n-tangent structure on $L_\pi E$, we will define this form directly in terms of invariant quantities on $L_\pi E$. We will first define CHP-1-forms, from which the CHP-m-form will be constructed.

The fundamental vertical vector fields E^{*i}_A are given in Lagrangian coordinates in (2.2). If $\mathcal{L} : L_\pi E \to \mathbb{R}$ is a lifted Lagrangian on $L_\pi E$, then it is the pull-up under ρ of a Lagrangian L on $J^1\pi$. Hence, since $\rho_*(\frac{\partial}{\partial u^A_i}) = \frac{\partial}{\partial y^A_i}$, we have

$$E^{*i}_A(\mathcal{L}) = u^i_k v^B_A \frac{\partial \mathcal{L}}{\partial u^B_k} = u^i_k v^B_A \frac{\partial \,\mathrm{L}}{\partial y^B_k}$$

This leads us to the following definition:

Definition 6.1 *Let $\mathcal{L} : L_\pi E \to \mathbb{R}$ be a Lagrangian on $L_\pi E$. The* **covariant momenta** *of $\mathcal{L}$ are*

$$\mathcal{P}^i_A = E^{*i}_A(\mathcal{L}) = (u^i_k v^B_A) p^k_B$$

where $p^k_B = \frac{\partial \mathcal{L}}{\partial u^B_k}$ denotes the **canonical momenta** *of the Lagrangian.*

Remark Notice that the **covariant momenta** $(\mathcal{P}^i_A)$ are globally defined tensorial objects on $L_\pi E$, while the **canonical momenta** $(p^k_B) = (\frac{\partial \mathcal{L}}{\partial u^B_k})$ are only defined locally.

We are now in a position to give a global definition of the CHP-form on $L_\pi E$. We first define the related 1-forms.

Definition 6.2 *Let $\mathcal{L} : L_\pi E \to \mathbb{R}$ be a lifted Lagrangian on $L_\pi E$, and $\tau(m)$ a positive function of m, the dimension of M. The CHP-1-forms $\theta^\alpha_{\mathcal{L}}$ on $L_\pi E$ are*

$$\theta^i_{\mathcal{L}} = \tau(m)\mathcal{L}\theta^i + E^{*i}_A(\mathcal{L})\theta^A \qquad (6.7)$$

$$\theta^A_{\mathcal{L}} = \theta^A \qquad (6.8)$$

Remark The positive function $\tau(m)$ in this definition is included to allow for various theories to occur as special cases. We will see that the choice $\tau(m) = 1$ yields the canonical theory of Carathéodory-Rund, and $\tau(m) = \frac{1}{m}$ yields the canonical theory of de Donder-Weyl.

Remark The collection of forms $(\theta^\alpha_{\mathcal{L}}) = (\theta^i_{\mathcal{L}}, \theta^A_{\mathcal{L}})$, where $\theta^A_{\mathcal{L}} = \theta^A$, is a **modified**, or **non-canonical soldering form** if $\mathcal{L} > 0$. This follows from the easily verifiable properties $X \lrcorner\ \theta^\alpha_{\mathcal{L}} = 0$ for all $\alpha = 1, 2, \ldots, n$ if and only if X is vertical with respect to $\lambda : L_\pi E \to E$ and $R^*_g \theta_{\mathcal{L}} = g^{-1} \cdot \theta_{\mathcal{L}}$.

Working out the local coordinate form of the CHP-1-forms in Lagrangian coordinates we find

$$\theta^i_{\mathcal{L}} = -H^i_j dx^j + P^i_A dy^A \tag{6.9}$$

$$\theta^A_{\mathcal{L}} = P^A_j dx^j + P^A_B dy^B \tag{6.10}$$

where

$$H^i_j = u^i_k (p^k_B u^B_j - \tau(m) \mathcal{L} \delta^k_j) \tag{6.11}$$

$$P^i_B = u^i_k p^k_B \tag{6.12}$$

$$P^A_j = -u^A_B u^B_j \tag{6.13}$$

$$P^A_B = u^A_B \tag{6.14}$$

We will refer to the H^i_j as the components of the **covariant Hamiltonian**, and to the P^i_B as the components of the **covariant canonical momentum**. If we define symbols h^k_j by the formula

$$h^k_j = p^k_B u^B_j - \tau(m) \mathcal{L} \delta^k_j \tag{6.15}$$

then the covariant Hamiltonian (6.11) can be expressed as $H^i_j = u^i_k h^k_j$. Setting $\tau(m) = 1$ we find that h^i_j has the form of Carathéodory's Hamiltonian [2, 16] tensor. Similarly, setting $\tau = \frac{1}{m}$ we find that $h = h^i_i$ yields the Hamiltonian in the de Donder-Weyl theory [4, 16].

Finally we show how the CHP-1-forms can be used to construct the CHP-m-form on $J^1\pi$.

Proposition 6.3 *Let (B_i, B_A) denote the standard horizontal vector fields of any torsion free linear connection on $\lambda : L_\pi E \to E$, and let vol denote the pull up to $L_\pi E$ of a fixed volume m-form on M. Set $vol_i = B_i \rfloor\ vol$. Then when $\tau(m) = \frac{1}{m}$ the m-form*

$$\theta_{\mathcal{L}} := \theta^i_{\mathcal{L}} \wedge vol_i$$

passes to the quotient to define the CHP-m-form $\Theta_{\mathcal{L}}$ on $J^1\pi$.

Proof The vector fields B_i have the local coordinate form $B_i = v^\alpha_i \frac{\partial}{\partial z^\alpha} + V$ where V is vertical with respect to $\lambda : L_\pi E \to E$. Using this and formulas (6.7) through (6.12) one can show that

$$\theta_{\mathcal{L}} = -(p^j_B u^B_j - \mathcal{L})\ vol + p^j_A dy^A \wedge (\frac{\partial}{\partial x^j} \rfloor\ vol)$$

The right-hand-side is constant on the fibers of $\rho : L_\pi E \to J^1\pi$ and is in fact the pull-up $\rho^*(\Theta_{\mathrm{L}})$ of the CHP-m-form Θ_{L} on $J^1\pi$. ■

Remark The above geometrical construction of the CHP-m-form is analogous to the geometrical construction given by Guillemin and Sternberg [8].

6.1 The Variational Principle on $L_\pi E$

We now lift the variational principle from $J^1\pi$ to $L_\pi E$. This is a simple procedure since we are using a lifted Lagrangian and only varying a section of π. Let $\phi : M \to E$ be a section of π and $j\phi$ its 1-jet prolongation to $J^1\pi$. For any section $\xi : J^1\pi \to L_\pi E$ we have that $u = \xi \circ j\phi : M \to L_\pi E$ is a section of $\pi \circ \lambda : L_\pi E \to M$.

The action integral on $J^1\pi$ lifts nicely since $\mathcal{L} = \mathrm{L} \circ \rho$.

$$\int_M j\phi^*(\mathrm{L}) vol \quad = \quad \int_M u^*(\mathcal{L}) vol$$

We recall [7] that the action integral is extremized by ϕ iff $j\phi^*(W \rfloor\ d\Theta_{\mathrm{L}}) = 0$ for all vector fields W on $J^1\pi$. However, this condition can be weakened to $j\phi^*(j(f) \rfloor\ d\Theta_{\mathrm{L}}) = 0$ for all vertical vector fields f on E.

Now for any such vertical vector field f, consider its $J^1\pi$ prolongation $j(f)$ and its n-symplectic Hamiltonian vector field $X_{\hat{f}}$ on $L_\pi E$ (also a kind of prolongation). From (4.2) we know $\rho_*(X_{\hat{f}}) = j(f)$. From proposition (6.3) we now have $X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}} = X_{\hat{f}} \lrcorner\, \rho^*(d\Theta_{\mathrm{L}})$. It follows that

$$u^*(X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}}) = j\phi^*(j(f) \lrcorner\, d\Theta_{\mathrm{L}})$$

We conclude that the action integral is extremized by ϕ iff $u^*(X_{\hat{f}} \lrcorner\, d\theta_{\mathcal{L}}) = 0$ for all vertical vector fields f on E.

7 The Generalized Hamilton-Jacobi Equation

As an application of our general formalism we derive the Carathéodory-Rund and de Donder-Weyl Hamilton-Jacobi equations. By analogy with the time independent Hamilton-Jacobi theory (see, for example, reference [18]) we seek Lagrangian submanifolds of $L_\pi E$. However, since the dimension of $L_\pi E$ is in general not twice the dimension of E, a new definition is needed. For our purposes here we will consider $n = m + k$ dimensional submanifolds of $L_\pi E$ that arise as sections of λ. In particular we consider sections $\sigma : E \rightarrow L_\pi E$ that satisfy

$$\sigma^*(d\theta^\alpha_{\mathcal{L}}) = 0 \tag{7.16}$$

We will refer to this equation as the *generalized Hamilton-Jacobi equation.*

Since $\sigma^*(d\theta^\alpha_{\mathcal{L}}) = d\left(\sigma^*(\theta^\alpha_{\mathcal{L}})\right)$ the condition (7.16) asserts that the 1-forms $\sigma^*(\theta^\alpha_{\mathcal{L}})$ are locally exact, and we express this as

$$\sigma^*(\theta^\alpha_{\mathcal{L}}) = dS^\alpha \tag{7.17}$$

in terms of $m + k$ new functions S^α defined on open subsets of E. For convenience we will denote objects on $L_\pi E$ pulled back to E using σ with an over-tilde. Thus, for example, $\tilde{H}^i_j = H^i_j \circ \sigma$ and $\tilde{P}^i_A = P^i_A \circ \sigma$. Then we get from (6.11)–(6.14) and (7.17)

$$\textbf{(a)}\ \ \tilde{H}^i_j = -\frac{\partial S^i}{\partial x^j}\ , \qquad \textbf{(b)}\ \ \tilde{P}^i_A = \frac{\partial S^i}{\partial y^A} \tag{7.18}$$

$$\textbf{(a)}\ \ \tilde{u}^A_B \tilde{u}^B_j = -\frac{\partial S^A}{\partial x^j}\ , \qquad \textbf{(b)}\ \ \tilde{u}^A_B = \frac{\partial S^A}{\partial y^B} \tag{7.19}$$

Recalling that $H^i_j = P^i_B u^B_j - \tau(m)\mathcal{L}u^i_j$ and P^i_A are functions of the coordinates x^i, y^A, u^i_j and u^A_i, equations (7.18) can be combined into the single equation

$$H^i_j(x^a, y^B, u^a_b, u^B_a, \frac{\partial S^i}{\partial y^B}) \circ \sigma = -\frac{\partial S^i}{\partial x^j} \tag{7.20}$$

Similarly combining equations (7.19) we obtain

$$\frac{dS^A}{dx^j} = 0$$

We next consider special cases of these **generalized Hamilton-Jacobi equations**.

7.1 The Theory of Carathéodory and Rund

We note from (6.11), (6.12), and (6.15) that $H^i_j = u^i_k h^k_j$ and $P^i_A = u^i_k p^k_A$, where the matrix of functions (u^i_j) is GL(m)-valued. Using the notation $P^i_j = -H^i_j$ and $\tilde{u}^i_j = u^i_j \circ \sigma$ we may rewrite (6.11) and (6.12) in the form

$$\tilde{P}^i_j = -\tilde{u}^i_k \tilde{h}^k_j \ , \qquad\qquad \tilde{P}^i_A = \tilde{u}^i_k \tilde{p}^k_A \tag{7.21}$$

If we take $t(m) = 1$ then these equations are the equations defining the *canonical momenta* in Rund's canonical formalism for Carathéodory's geodesic field theory (see equations (1.22), page 389 in [16], with the obvious change in notation). In this situation equation (7.20) can be identified with the Rund's Hamilton-Jacobi equation for Carathéodory's theory (see equation (3.29) on page 240 in [16]). We recall [16] that one can derive the Euler-Lagrange field equations from this Hamilton-Jacobi equation.

In (7.21) we have the result that the arbitrary non-singular matrix-valued functions $(\tilde{u}^i_j)$ that occur in Rund's canonical formalism for Carathéodory's theory have a geometrical interpretation in the present setting. Specifically they correspond to the coordinates for linear frames for M. These defining relations are derived from Rund's **transversality condition**, and this condition has the elegant reformulation here as the kernel of $(\theta^i_{\mathcal{L}})$.

We will say that a vector X at $e \in E$ is transverse to a solution surface through e that is defined by a given Lagrangian $\mathcal{L}$, if $X = d\lambda(\hat{X})$, where $\hat{X} \in T_u(L_\pi E)$ satisfies $\hat{X} \,\lrcorner\, \theta^i_{\mathcal{L}} = 0$, for some $u \in \lambda^{-1}(e)$. $\hat{X}$ thus satisfies the equations

$$0 = -H^i_j X^j + P^i_A X^A = u^i_k \left(-h^k_j X^j + p^k_A X^A \right)$$

$$X^j = \hat{X}(x^i) \ , \quad X^A = \hat{X}(y^A)$$

from which we infer

$$0 = -h^k_j X^j + p^k_A X^A \tag{7.22}$$

This is Rund's transversality condition for the theory of Carathéodory when we take $t(m) = 1$ (see equation (1.10), page 388 in [16]). The canonical momenta P^i_j and P^i_A are defined by Rund to be solutions of

$$0 = P^i_j X^j + P^i_A X^A \tag{7.23}$$

when (X^j, X^A) satisfy (7.22). Rund's solutions of these equations are given in (7.21). Looking at (7.21), (7.22) and (7.23) we see that the introduction of the u^i_j in (7.21) amounts to the introduction of the $\mathrm{GL}(m)$ freedom for linear frames for M.

7.2 de Donder-Weyl Theory

Returning to (7.20) let us reduce this equation by making several assumptions. We suppose that $\mathcal{L}$ is regular (in the usual sense on $J^1\pi$), that the section σ is such that $\tilde{u}^i_j = \delta^i_j$, and we make the choice $t(m) = \frac{1}{m}$. Now summing $i = j$ in (7.20) we obtain

$$\tilde{h}(x^i, y^B, \frac{\partial S^i}{\partial y^B}) = -\frac{\partial S^i}{\partial x^i}$$

where $\tilde{h} = \tilde{p}^i_A \tilde{u}^A_i - \tilde{\mathcal{L}}$. This equation is the Hamilton-Jacobi equation of the de Donder-Weyl theory, as presented by Rund (see equation (2.31) on page 224 in [16]). We recall [16] that one can derive in this case also the Euler-Lagrange field equations from the de Donder-Weyl Hamilton-Jacobi equation.

8 Hamilton's Equations

The structure of equations (6.9) - (6.12) suggests that one should be able to derive generalized Hamilton equations if the canonical momenta $p^i_A = \frac{\partial \mathcal{L}}{\partial u^A_i}$ can be introduced as part of a local coordinate system on $L_\pi E$. Part of the original philosophy used in developing n-symplectic geometry in reference [12] was to switch from scalar equations to tensor equations, motivated by the fact that the soldering 1-form is vector-valued. In particular, the basic structure equation (2.4) in n-symplectic geometry is tensor-valued. We show next that

$$u^*(\eta \lrcorner\ d\theta^i_{\mathcal{L}}) = 0 \tag{8.24}$$

where $u : M \to L_\pi E$ is a section of $\pi \circ \lambda$, and η is any vector field on $L_\pi E$, yields generalized canonical equations that contain known canonical equations as special cases. We consider here only $d\theta^i_{\mathcal{L}}$ since by Proposition (6.3) it alone is needed to construct the CHP-m-form on $J^1\pi$.

We need the following definition in order to introduce the canonical momenta as part of a coordinate system on $L_\pi E$.

Definition 8.1 *A Lagrangian $\mathcal{L}$ on $L_\pi E$ is* **regular** *if the $(m+k) \times (m+k)$ matrix*

$$\left(E^{*i}_A \circ E^{*j}_B(\mathcal{L})\right)$$

is non-singular.

Working out the terms of this matrix in Lagrangian coordinates using (2.2) we obtain

$$E^{*i}_A \circ E^{*j}_B(\mathcal{L}) = u^j_a u^i_b v^E_B v^D_A \left(\frac{\partial^2 \mathcal{L}}{\partial u^E_a \partial u^D_b}\right)$$

It is clear that this definition is equivalent to the standard definition of regularity on $J^1\pi$.

We now consider the transformation of coordinates from the set $(x^i, y^A, u^i_j, u^A_k, u^A_B)$ to the new set $(\bar{x}^i, \bar{y}^A, \bar{u}^i_j, p^j_A, \bar{u}^A_B)$ where

$$\bar{x}^i = x^i \ , \quad \bar{y}^A = y^A \ , \quad \bar{u}^i_j = u^i_j \ , \quad \bar{u}^A_B = u^A_B \ , \quad p^i_A = \frac{\partial \mathcal{L}}{\partial u^A_i}$$

Computing the Jacobian one finds that the new barred functions will be a proper coordinate system whenever the Lagrangian is regular. For the remainder of this section we shall assume that $\mathcal{L}$ has this property, despite the fact that many important examples (see [7]) have non-regular Lagrangians. Moreover, for simplicity we will drop the bars on the new coordinates.

In the generalized canonical equation (8.24) we now take $\eta = \frac{\partial}{\partial p^i_A}$. We find the result

$$0 = \left(\frac{\partial H^j_k}{\partial p^i_A} \circ u\right) + (u^j_i \circ u)\left(\frac{\partial (y^A \circ u)}{\partial x^k}\right)$$

Using $H^j_k = u^j_i h^i_k$ and the fact that (u^i_j) is a non-singular matrix valued function, this last equation reduces to

$$\frac{\partial h^j_k}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^k} \delta^j_i$$

This is our first set of **generalized Hamilton equations**. Notice that by summing $j = k$ in this equation we obtain

$$\frac{\partial h}{\partial p^i_A} \circ u = \frac{\partial (y^A \circ u)}{\partial x^i} \tag{8.25}$$

Upon setting $t(m) = \frac{1}{m}$ we obtain half of the de Donder-Weyl canonical equations. Under suitable but complicated conditions these equations, with $t(m) = 1$, will also reproduce part of Rund's canonical equations for the theory of Carathéodory.

In the generalized canonical equation (8.24) we now take $\eta = \frac{\partial}{\partial y^A}$. We find

$$0 = u^* \left(d(u^i_k p^k_A) + u^i_k \frac{\partial h^k_j}{\partial y^A} dx^j \right)$$

Using an "over bar" notation to denote objects pulled back to M by u we may write this as

$$\frac{\partial}{\partial x^j} \left(\bar{u}^i_k \bar{p}^k_A \right) = -\bar{u}^i_k \left(\frac{\partial h^k_j}{\partial y^A} \right) \circ u \tag{8.26}$$

This is our second set of **generalized Hamilton's equations**.

Notice that what is non-standard in (8.26) is the appearance of the derivatives of the functions $\bar{u}^i_j = u^i_j \circ u$. If, however, the section $u : M \to L_\pi E$ is such that the $\bar{u}^i_j$ are constants, then these equations reduce to

$$\frac{\partial (\bar{p}^k_A)}{\partial x^j} = -\frac{\partial h^k_j}{\partial y^A} \circ u$$

Setting $\tau(m) = \frac{1}{m}$ and summing $k = j$ in this equation we obtain

$$\frac{\partial (\bar{p}^i_A)}{\partial x^i} = -\frac{\partial h}{\partial y^A} \circ u$$

These equations, together with equations (8.25) when $\tau(m) = \frac{1}{m}$, are the complete canonical equations in the de Donder-Weyl theory.

9 A Model Theory on $L_\pi E$

Equation (8.26) clearly suggests that what is needed in this new formalism on $L_\pi E$ are dynamical equations for the coordinates u^i_j of frames for the parameter space M. We note that our fundamental canonical variables $P^i_j = -H^i_j$ and P^i_A on $L_\pi E$ are functions of the variables (x^i, y^A, u^i_j, u^A_i), but they do not depend on the other vertical coordinates u^A_B. On the other hand the complete set of momentum variables $(P^i_j, P^i_A, P^A_i, P^A_B)$ defined in (6.11)–(6.14) depend on all the coordinates on $L_\pi E$. It is evident that a complete Lagrangian field theory on $L_\pi E$ would thus need to supply field equations for the coordinates u^i_j of frames for M as well as equations for the coordinates u^A_B of vertical frames for E, in addition to the standard field equations for sections of π. To do this in as simple a way as possible we introduce the following total Lagrangian:

$$\mathcal{L}_{\mathrm{TOTAL}} = \mathcal{L}_E + \mathcal{L}_{M,V}$$

Here $\mathcal{L}_{M,V}$ is a Lagrangian for the ρ-vertical variables u^i_j and u^A_B, and $\mathcal{L}_E$ is to be a generalization of $\rho^*(\mathrm{L})$ that now includes a coupling to the vertical coordinates.

For an example suppose that $\pi : E \to M$ is a vector bundle over spacetime M. We can take $\mathcal{L}_{M,V}$ to be the Lagrangian for a higher-dimensional Kaluza-Klein metric tensor on E, but written in n-tuple form using u^α_β. The term $\mathcal{L}_E$ can then be taken to be the usual Lagrangian for a section of π, but with the usual "fixed" metric tensors now replaced by the dynamical metric written in terms of n-tuples.

As a second example suppose that $\pi : E \to M$ is a principal bundle over spacetime M. Then one can take $\mathcal{L}_{TOTAL}$ to be a Lagrangian for the Yang-Mills generalization of the Kaluza-Klein theory as formulated, for example, by Hermann [9].

We note that sections of LM may be considered as 1-jets of non-singular maps from $\mathbb{R}^m \to M$ since $LM \subset J^1(\mathbb{R}^m, M)$. Similarly since $LV \subset J^1(\mathbb{R}^k, E)$, sections of LV may be considered as 1-jets of maps from $\mathbb{R}^k \to E$. From a jet bundle point of view we may therefore consider the total Lagrangian $\mathcal{L}_{\mathrm{TOTAL}}$ as defined on a subset of the space

$$J^1\pi \times_E (J^1(\mathbb{R}^k, E) \times_M J^1(\mathbb{R}^m, M))$$

It seems clear then that $L_\pi E$ is a natural arena for unified field theories in which the determination of a geometry for M, and a geometry for the fibers of E, are both part of the dynamical problem. Field equations would produce, for our examples given above, a metric of the Kaluza-Klein type for tangents to E together with a field (section of π) defined on M. If one fixes the M and V gauges by reducing H to $\{ I \}$ and thereby reducing $\mathcal{L}_{TOTAL}$ to $\rho^*(\mathrm{L})$, then one arrives at the generalized Lagrangian field theory on $J^1\pi$ lifted to $L_\pi E$ that was discussed earlier.

10 Conclusions

In this paper we have reformulated covariant field theory for sections of $\pi : E \to M$ on the bundle of vertically adapted linear frames $L_\pi E$. The advantaged gained by this reformulation is that it allows us to utilize the natural geometry that is supported on $L_\pi E$, namely n-symplectic geometry, to further develop covariant field theory. We have concentrated on demonstrating that $L_\pi E$ with its canonical n-symplectic structure provides an appropriate arena for formulating covariant field theory, leaving aside the development of the modified n-symplectic geometry defined by the Cartan-Hamilton-Poincaré (CHP) 1-forms to future papers.

To this end we showed that covariant field theory on $J^1\pi$ lifts in a natural way to $L_\pi E$. The analysis was based on the theorem, presented in section 3, that $J^1\pi$ is a principal fiber bundle $\rho : L_\pi E \to J^1\pi$ over the bundle of 1-jets of sections of π. This theorem was used to show that the soldering 1-forms θ^α on $L_\pi E$ play the role of the contact structure on $J^1\pi$. The soldering 1-forms and a lifted Lagrangian $\mathcal{L} = \rho^*(\mathrm{L})$ were then used to constructed modified soldering 1-forms $\theta^\alpha_{\mathcal{L}}$ on $L_\pi E$, the CHP 1-forms. These CHP 1-forms were shown to pass to the quotient to define the standard CHP m-form on $J^1\pi$. Further we used the CHP 1-forms to derive generalized Hamilton-Jacobi and generalized canonical equations on $L_\pi E$, and then showed that the Hamilton-Jacobi and canonical equations in the theories of Carathéodory-Rund and de Donder-Weyl are contained as special cases. What we did not do was develop the explicit structure of the modified n-symplectic geometry, including allowable observables, Hamiltonian vector fields, and Poisson and graded Poisson brackets, that one should be able to define using the CHP 1-forms. Nor did we develop the variational principle on $L_\pi E$ for a Lagrangian that is not lifted from $J^1\pi$. These problems we leave to future publications.

As we studied the structure of $L_\pi E$ it became apparent that the extra degrees of freedom in $L_\pi E$ that are not in $J^1\pi$ need not be ghost degrees of freedom, but may have direct physical significance. Part of the extra degrees of freedom, namely coordinates for the frames of M, were identified with the undetermined elements in the Carathéodory-Rund canonical formalism. The manifold $L_\pi E$ emerged as a natural arena for a unified theory that contains in addition to the sector for sections of π, dynamical sectors for a geometry for M and a geometry for the fibers of E. In fact it was shown that the manifold $L_\pi E$ is a subset of the jet space $J^1\pi \times_E (J^1(\mathbb{R}^k, E) \times_M J^1(\mathbb{R}^m, M))$. We argued in section 9 that one may easily write down on $L_\pi E$ a model Lagrangian, in n-tuple form, for a type of higher dimensional Kaluza-Klein theory. We intend to pursue these ideas in future work.

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