On the Affine Connection Structure of the Charged Symplectic 2-Form†

L. K. Norris‡

ABSTRACT

It is shown that the charged symplectic form in Hamiltonian dynamics of classical charged particles in electromagnetic fields defines a generalized affine connection on an affine frame bundle associated with spacetime. Conversely, a generalized affine connection can be used to construct a symplectic 2-form if the associated linear connection is torsion–free and the anti-symmetric part of the R^{4*} translational connection is locally derivable from a potential. Hamiltonian dynamics for classical charged particles in combined gravitational and electromagnetic fields can therefore be reformulated as a $P(4) = O(1,3) \otimes R^{4*}$ geometric theory with phase space the affine cotangent bundle AT^*M of spacetime. The source-free Maxwell equations are reformulated as a pair of geometrical conditions on the $\mathbb{R}^{4*}$ curvature that are exactly analogous to the source-free Einstein equations.

†International Journal of Theoretical Physics, **30**, pp. 1127-1150 (1991)

‡Department of Mathematics, Box 8205, North Carolina State University, Raleigh, NC 27695-8205

1. Introduction

The problem of geometrizing the relativistic classical mechanics of charged test particles in curved spacetime is closely related to the larger problem of finding a geometrical unification of the gravitational and electromagnetic fields. In a geometrically unified theory one would expect the equations of motion of classical charged test particles to be fundamental to the geometry in a way analogous to the way uncharged test particle trajectories are geometrized as linear geodesics in general relativity. Since a satisfactory unified theory should contain the known observational laws of mechanics in some appropriate limit, one can gain insight into the larger unification problem by analyzing the geometrical foundations of classical mechanics.

This paper is concerned with the question of the geometrical unification of the gravitational and electromagentic fields, and accordingly we analyze the geometry of Hamiltonian mechanics of classical charged particles in electromagnetic fields. We show that the usual formulation in terms of symplectic geometry on the momentum-energy phase space T^*M defines a $P(4) = O(1,3) \otimes R^{4*}$ generalized affine connection on an affine frame bundle AM of spacetime M. The resulting affine geometry on spacetime is the geometry of the recently proposed P(4) geometrical theory of gravitation and electromagnetism (Norris, 1985; Kheyfets and Norris, 1988). The new features of the P(4) theory are that the gauge group associated with the electromagnetic field is the group R^{4*} of momentum-energy translations, with the Maxwell field tensor playing the role of the $\mathbb{R}^{4*}$ gauge potential, and the momentum-energy phase space is the affine cotangent bundle AT^*M.

There are two standard ways to formulate canonical mechanics for a classical charged particle in an electromagnetic field in spacetime. The more familiar method uses standard Poisson brackets (i.e. the canonical symplectic 2-form on phase space T^*M) and the generalized momentum-energy $\pi_\mu = mg_{\mu\nu}\dot{x}^\nu + eA_\mu$. Because the electromagnetic vector potential A_μ occurs explicitly in the definition of π_μ the generalized momentum-energy is not gauge invariant, and this leads to difficulties in physical interpretation. However, no such problem arises with the spacetime equations of motion derived from the canonical equations because they involve only the gauge invariant field strengths $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

To avoid the difficulties with interpretation one may use an alternative method (Torrence and Tulczyjew, 1973; Sniatycki, 1974;Woodhouse, 1980) in which the momentum-energy variable π_μ is the gauge-invariant kinetic momentum–energy $mg_{\mu\nu}\dot{x}^\nu$, but then one must also use nonstandard Poisson brackets (i.e. the "charged" symplectic 2-form). In addition this alternative approach employs the "free-particle" Hamiltonian $\mathcal{H} = \frac{1}{2m}(m^2 + g^{\mu\nu}\pi_\mu\pi_\nu)$ even though the particle in question is a charged particle.

The transformation between these two formulations is the **non canonical** transformation $\pi_\mu \longrightarrow \tilde{\pi}_\mu = \pi_\mu + eA_\mu$, which is the well-known "substitution rule" of elementary mechanics. When this transformation is treated as a momentum-energy translation one must generalize the usual definition of phase space coordinates in terms of linear frames and use affine frames instead. This leads from $Gl(4)$ covariance to $A(4) = Gl(4) \otimes R^{4*}$ covariance and means that the bundle of linear frames, the geometrical arena for the linear Riemannian geometry of general relativity, is replaced by the bundle of affine frames. A generalized affine connection on AM may be thought of (Kobayashi and Nomizu, 1963) as composed of a pair (Γ, K) where Γ is a linear connection and K, a (0,2) tensor field on spacetime, represents the R^{4*} translational part of the connection. We show that the charged symplectic form defines an affine connection on AM with the electromagnetic field tensor playing the role of the R^{4*} part of a P(4) affine connection, while the linear geometry is the Riemannian geometry of spacetime.

In Section 2 we show how the charged symplectic form S_c is related to momentum-energy translations, and how one may extract from S_c a definition of a vector bundle affine connection on T^*M. Then in Section 3 vector bundle affine connections are related to affine connections on AM. In the process we find that it is natural to generalize the phase space of classical charged particles in electromagnetic fields from T^*M to the affine cotangent bundle AT^*M.

In Section 4 we reverse the process and find necessary and sufficient conditions for an arbitrary affine connection on AM to define a symplectic structure on AT^*M. The conditions are: (1) the associated Gl(4) linear connection must be torsion free, and (2) the skew–symmetric part of the R^{4*} translational connection must be locally derivable from a potential.

In Section 5 we use the results developed in earlier sections to reinterpret the canonical mechanics of charged particles on AT^*M in terms of P(4) affine geometry on AM. We first show that the Hamilton equations of motion on spacetime, the Lorentz force law, can be reinterpreted as the equation of an affine geodesic on M with respect to the natural affine connection on AM induced by the charged symplectic form. We then complete the reinterpretation by showing that the Maxwell equations for the source–free electromagnetic field have a natural geometrical formulation in terms of the R^4–curvature tensor that is remarkably parallel to the geometric vacuum Einstein equations. In an appendix we provide the basic material on the affine frame bundle and related associated bundles that is needed in Section 3.

The standard affine frame bundle AM of a manifold M consists of all triples (p, e_μ, t) , where $p \in M$, (e_μ) is a linear frame at p, and t is a tangent vector at p (Kobayashi and Nomizu, 1963). The structure group of AM is the affine group $A(4) = Gl(4) \otimes R^4$ with group multiplication

$$(A_1, \xi_1) \cdot (A_2, \xi_2) = (A_1 A_2, A_1 \cdot \xi_2 + \xi_1) \qquad , \forall \ (A_1, \xi_1), (A_2, \xi_2) \in A(4) \ .$$

In this paper we consider the momentum–energy of a particle as being a **covariant** quantity, and in order to deal with affine covariant vector fields on a manifold in a natural way we will consider in place of AM a modified affine frame bundle $\hat{A}M$. The points of $\hat{A}M$ are triples (p, e_μ, β) , where $p \in M$, (e_μ) is a linear frame at p, and β is a **covector** at p. The structure group of $\hat{A}M$ is the affine group $\hat{A}(4) = Gl(4) \otimes R^{4*}$ with group multiplication

$$(A_1, \xi_1) \cdot (A_2, \xi_2) = (A_1 A_2, \xi_1 \cdot A_2 + \xi_2) \qquad , \forall \ (A_1, \xi_1), (A_2, \xi_2) \in \hat{A}(4) \ .$$

$\hat{A}(4)$ is isomorphic to the opposite group of $A(4)$. Both AM and $\hat{A}M$ contain the linear frame bundle of M as subbundles, so that the linear differential geometry of a manifold may be described using either affine bundle. In order to deal with affine **covariant** vector fields in an efficient way we will work with $\hat{A}M$ rather than with AM. However, in order to simplify notation we will refer to $\hat{A}M$ as the affine frame bundle of M and denote it simply by AM. Similarly we will write $A(4)$ for $\hat{A}(4)$ and denote by P(4) the Poincarè subgroup $O(1,3) \otimes R^{4*}$ of $\hat{A}(4)$.

2. The R^{4*} Affine Connection Defined by the Charged Symplectic 2-form

The momentum-energy phase space for Hamiltonian dynamics of a single particle in flat spacetime (M,g) is the cotangent bundle $T^*M \overset{proj}{\longrightarrow} M$. $proj$ is the projection map $proj(p,\beta) = p$ for β a covector at $p \in M$. Coordinates $y^i = (q^\mu, \pi_\nu),\ i = 1,\ldots,8\ ,\mu,\nu = 1,\ldots,4$, on T^*M are standardly defined in terms of coordinates x^μ and the associated linear frame field $(e_\mu) = (\frac{\partial}{\partial x^\mu})$ on M by

$$\begin{aligned}(y^i)(p,\beta) &= (q^\mu,\pi_\nu)(p,\beta)\\ &= (x^\mu(p), \beta(e_\nu(p)))\ .\end{aligned} \tag{1}$$

In this definition $q^\mu = (proj)^*(x^\mu)$, while the vertical coordinates π_ν are the real-valued functions on T^*M defined by

$$\pi_\nu(p,\beta) := \beta(e_\nu(p)) \tag{2}$$

for $p \to \beta(p)$ a section of T^*M.

The **canonical symplectic form** on T^*M is the 2-form $S := d\theta$ where θ is the **canonical 1-form** defined invariantly by $\theta_{(p,\beta)}(X) = \beta((proj_* X)(p))$ for X a vector field on T^*M. In the local coordinates (q^μ, π_ν) S takes the form

$$S = d\pi_\mu \wedge dq^\mu \quad . \tag{3}$$

The free particle dynamical system is defined by S and the Hamiltonian $\mathcal{H} : T^*M \to R$ given by

$$\mathcal{H}(q^\mu,\pi_\nu) := \frac{1}{2m}[m^2 + g^{\mu\nu}\pi_\mu\pi_\nu] \quad . \tag{4}$$

The $g^{\mu\nu}$ in (4) are the components of the spacetime metric tensor and take the form $diag(-1,1,1,1)$ in flat spacetime when the spacetime coordinates x^μ are Lorentzian coordinates and the linear frame (e_ν) is an orthonormal (O.N.) linear frame field on M.

The Hamiltonian vector field $X_{\mathcal{H}}$ on T^*M determined by a Hamiltonian $\mathcal{H}$ is the unique solution of the equation

$$d\mathcal{H} = -X_{\mathcal{H}} \lrcorner\ S \quad . \tag{5}$$

The hook product is defined for a 2-form $\omega \wedge \lambda$ and a vector field X by $X \lrcorner\ (\omega \wedge \lambda) = \omega(X)\lambda - \lambda(X)\omega$.

The differential equations for the integral curves of $X_{\mathcal{H}}$ determined by (4) are the pair of Hamilton equations

$$\dot{q}^\mu = \frac{\partial \mathcal{H}}{\partial \pi_\mu} = \frac{\pi_\nu g^{\nu\mu}}{m} \quad , \tag{6-a}$$

$$\dot{\pi}_\nu = -\frac{\partial \mathcal{H}}{\partial q^\nu} = 0 \quad . \tag{6-b}$$

Combining these phase space equations in the usual way leads to the spacetime free particle equations of motion

$$\ddot{x}^\mu = 0 \quad . \tag{7}$$

The standard prescription for introducing the electromagnetic interaction is to introduce the electromagnetic vector potential $A = A_\mu e^\mu$ via the "substitution rule"

$$\pi_\nu \longrightarrow \tilde{\pi}_\nu = \pi_\nu + e\hat{A}_\nu \qquad , \quad \hat{A}_\nu := proj^*(A_\nu) \quad . \tag{8}$$

On phase space T^*M this may be considered as the coordinate transformation

$$(q^\mu, \pi_\nu) \longrightarrow (\tilde{q}^\mu, \tilde{\pi}_\nu) = (q^\mu, \pi_\nu + e\hat{A}_\nu) \quad . \tag{9}$$

This transformation is clearly a vertical translation along the fibers of T^*M, and it is incompatible with the definition of coordinates given above. Recall that $\pi_\nu(p,\beta) = \beta(e_\nu(p))$ where (e_ν) is a linear frame field. Under change of linear frame $(e_\mu) \to (\tilde{e}_\mu) = (e_\lambda a^\lambda_\mu)$ with $(a^\lambda_\mu) \in O(1,3)$ the coordinates $y^{\nu+4} = \pi_\nu$ undergo linear **homogeneous** transformations, while (9) is **inhomogeneous**.

To allow for the translations (9) we use affine frames (see the Appendix) to generalize the definition of the coordinates π_ν. An affine frame field on M is denoted by (e_μ, t) where (e_μ) is an O.N. frame field and t is a covector field on M, the **origin** of the affine frame.

Define affine coordinates y^i , $i = 1, \ldots, 4$, by $q^\mu = (proj)^*(x^\mu)$ as in (1), and for $i = 5, \ldots, 8$ by

$$\begin{aligned} \pi_\mu(p, \beta) &= (\beta - t(p))(e_\mu(p)) \\ &= \beta(e_\mu(p)) - t_\mu(p) \quad . \end{aligned} \tag{10}$$

Let (e^μ) denote the coframe dual to (e_μ) . Then under the change of affine frame

$$(e_\mu, t) \longrightarrow (\tilde{e}_\mu, \tilde{t}) = (e_\lambda a^\lambda_\mu, t - \xi_\lambda (a^{-1})^\lambda_\nu e^\nu) \quad , \quad (a^\lambda_\mu, \xi_\lambda) \in P(4) \quad , \tag{11-a}$$

the transformation law for coordinates (q^μ, π_ν) is

$$(q^\mu, \pi_\nu) \longrightarrow (\tilde{q}^\mu, \tilde{\pi}_\nu) = ((a^{-1})^\mu_\lambda q^\lambda, a^\lambda_\nu \pi_\lambda + \xi_\nu) \quad . \tag{11-b}$$

The coordinate translations (9) are well-defined with respect to these affine coordinates. In particular the coordinates $(\tilde{q}^\mu, \tilde{\pi}_\nu) = (q^\mu, \pi_\nu + e\hat{A}_\nu)$ in (9) are defined by the affine frame (e_μ, eA_ν) if (q^μ, π_ν) are defined by $(e_\mu, 0)$. In Section 3 we will see that this generalization means that we have in fact replaced T^*M with AT^*M, the affine cotangent bundle.

Transforming the Hamiltonian $\mathcal{H}$ given in equation (4) to new coordinates $(\tilde{q}^\mu, \tilde{\pi}_\nu)$ using the momentum-energy translation (9) we get

$$\begin{aligned} \mathcal{H}(\tilde{q}^\mu, \tilde{\pi}_\nu - e\hat{A}_\nu) &= \tilde{\mathcal{H}}(\tilde{q}^\mu, \tilde{\pi}_\nu) \\ &= \frac{1}{2m}[m^2 + g^{\mu\nu}(\tilde{\pi}_\mu - e\hat{A}_\mu)(\tilde{\pi}_\nu - e\hat{A}_\nu)] \quad . \end{aligned} \tag{13}$$

We obtain this Hamiltonian whether we use the "substitution rule" or the coordinate transformation interpretation.

To complete the charged particle system we must choose a new sympletic 2-form $\tilde{S}$ defined by

$$\tilde{S}(\tilde{q}^\mu, \tilde{\pi}_\nu) = d\tilde{\pi}_\mu \wedge dq^\mu \quad . \tag{14}$$

The 2-form $\tilde{S}$ is one representation of the " charged" symplectic form (c.f. equation (21)) (Torrence and Tulczyjew, 1973; Sniatycki, 1974;Woodhouse, 1980).

The equations that follow from

$$d\tilde{\mathcal{H}} = -X_{\tilde{\mathcal{H}}} \rfloor \tilde{S} \tag{15}$$

are now the well-known Hamilton equations

$$\begin{aligned} \frac{d}{ds}(\tilde{q}^\mu) &= \frac{1}{m} g^{\mu\nu}(\tilde{\pi}_\nu - e\hat{A}_\nu) \quad , \\ \frac{d}{ds}(\tilde{\pi}_\nu) &= \frac{e}{m}(\partial_\nu \hat{A}^\lambda)(\tilde{\pi}_\lambda - e\hat{A}_\lambda) \quad . \end{aligned} \tag{16}$$

These equations combine to give the Lorentz force law

$$\ddot{x}^\mu = \frac{e}{m} F^\mu{}_\lambda \dot{x}^\lambda \tag{17}$$

on spacetime, where $F^\mu{}_\lambda = \nabla^\mu A_\lambda - \nabla_\lambda A^\mu$ is the (1,1) form of the Maxwell field tensor.

We observe that the choice (14) means that the coordinates $(\tilde{q}^\mu, \tilde{\pi}_\nu)$ are canonical coordinates for the charged particle system. Had we not introduced the new symplectic form $\tilde{S}$, but merely transformed S to new coordinates using (9) as we did with $\mathcal{H}$, we would have found ($\hat{F} := proj^*(F)$)

$$\begin{aligned} S(q,\pi) &= S(q, \tilde{\pi} - e\hat{A}) \\ &= d\tilde{\pi}_\mu \wedge dq^\mu - \frac{e}{2}\hat{F}_{\mu\nu} dq^\mu \wedge dq^\nu \\ &= \tilde{S} - \frac{e}{2}\hat{F} \quad . \end{aligned} \tag{18}$$

Thus the coordinates $(\tilde{q}^\mu, \tilde{\pi}_\nu) = (q^\mu, \tilde{\pi}_\nu)$ are non-canonical with respect to S. Note that since the Hamiltonian equations are coordinate independent, (13) and (18) would lead to equation (7) rather than equation (17).

Since we now have the momentum-energy phase space coordinates tied to affine frames we can transfer characteristic properties from the coordinates to the affine frames that define them. We will refer to (e_μ, eA) as a canonical affine frame and to $(e_\mu, 0)$ as a non-canonical affine frame (Kheyfets and Norris, 1988) for the charged particle system.

We can now transform back to the non-canonical coordinates (q^μ, π_ν) defined by $(e_\mu, 0)$ using the P(4) affine transformation inverse to (9), namely

$$\tilde{\pi}_\mu \longrightarrow \pi_\mu = \tilde{\pi}_\mu - e\hat{A}_\mu \quad . \tag{19}$$

Using (19) in (13) and (14) we find

$$\begin{aligned} \tilde{\mathcal{H}}(\tilde{q}, \tilde{\pi}) &= \tilde{\mathcal{H}}(q, \pi + eA) \\ &= \mathcal{H}(q, \pi) \\ &= \frac{1}{2m}[m^2 + g^{\mu\nu}\pi_\mu\pi_\nu] \quad , \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
\tilde{S}(\tilde{q},\tilde{\pi}) &= \tilde{S}(q,\pi+eA) \\
&= S + \frac{e}{2}\hat{F} \\
&= d\pi_\mu \wedge dq^\mu + \frac{e}{2}\hat{F}_{\mu\nu} dq^\mu \wedge dq^\nu \quad .
\end{aligned} \tag{21}
$$

This two-form $\tilde{S}$ is the "charged" symplectic form (Torrence and Tulczyjew, 1973; Sniatycki, 1974;Woodhouse, 1980). .

Thus although the coordinates (q^μ, π_ν) defined by $(e_\mu, 0)$ are non-canonical for the charged particle system, the Hamiltonian is in free particle form when espressed in terms of them. An interpretation has been given (Kheyfets and Norris, 1988) of this free particle Hamiltonian in terms of the instantaneously comoving inertial frames used in the operational definition of the Lorentz force law.

We have the situation that $\tilde{S}$ is in canonical form (14) relative to (e_μ, eA) but in non-canonical form (21) relative to $(e_\mu, 0)$. Define 1-forms C_μ on T^*M by

$$
C_\mu := d\pi_\mu - \frac{e}{2}\hat{F}_{\mu\nu} dq^\nu \quad . \tag{22}
$$

The symplectic 2-form $\tilde{S}$ can now be expressed relative to (q^μ, π_ν) as

$$
\tilde{S} = C_\mu \wedge dq^\mu \quad . \tag{23}
$$

We will see below that the C_μ define an R^{4*} affine connection. Accordingly we can refer to coordinates (q^μ, π_ν) as **covariant canonical coordinates**, the covariance referring to P(4) transformations (11).

Since dF = 0 for a Maxwell field we verify easily that $\tilde{S}$ is closed:

$$
d\tilde{S} = dC_\mu \wedge dq^\mu = \frac{e}{2} d\hat{F} = 0 \quad . \tag{24}
$$

However $dC_\mu \neq 0$ generally since

$$
dC_\mu = \frac{e}{2}\partial_{[\alpha}\hat{F}_{\beta]\mu} dq^\alpha \wedge dq^\beta \quad . \tag{25}
$$

The 1-forms C_μ are exact if and only if the electromagnetic field is covariant constant (static and uniform) on Minkowski spacetime (Norris, 1985).

The significance of (22) is due to the following theorem (Hermann, 1975). Let θ_μ denote 1-forms defining an Ehresmann connection on the vector bundle T^*M. In an affine coordinate system (q^μ, π_ν) these 1- forms have the general form

$$\theta_\mu = d\pi_\mu - f_{\mu\nu} dq^\nu \tag{26}$$

where the $f_{\mu\nu}$ are 16 arbitrary functions on T^*M. In the theorem that follows an affine connection on T^*M is an Ehresmann connection that defines affine (i.e. linear, inhomogeneous) maps between the fibers of T^*M by parallel transport.

Theorem:(Hermann,1975) The functions $f_{\mu\nu}$ on T^*M determine an affine connection for $T^*M \stackrel{proj}{\longrightarrow} M$ if and only if they are of the inhomogeneous-linear form

$$f_{\mu\nu} = B_{\mu\nu}(q) + B^\lambda_{\mu\nu}(q)\pi_\lambda \tag{27}$$

where the $(B_{\mu\nu}, B^\lambda_{\mu\nu})$ are pull backs of functions on M, which uniquely determine the affine connection. Conversely, such functions can be given arbitrarily, and then determine an affine connection.

Introduce connection 1-forms ω_μ and ${\omega^\nu}_\mu$ by

$$\begin{aligned} \omega_\mu &:= B_{\nu\mu} dq^\nu \quad , \\ {\omega^\nu}_\mu &:= B^\nu_{\lambda\mu} dq^\lambda \quad . \end{aligned} \tag{28}$$

From (26) - (28) we get for the general form of affine connection 1-forms on T^*M

$$\theta_\mu = d\pi_\mu - \omega_\mu - {\omega^\nu}_\mu \pi_\nu \quad , \tag{29}$$

where ω_μ and ${\omega^\nu}_\mu$ are pull backs under *proj* of 1-forms on M. Comparing (22) with (29) using (28) we find that the C_μ define an affine connection on T^*M with

$$\omega_\mu = \frac{e}{2} \hat{F}_{\mu\nu} dq^\nu \quad , \tag{30}$$

$${\omega^\nu}_\mu = 0 \quad . \tag{31}$$

3. Generalized Affine Connections on AM

The affine connection just described is a vector bundle affine connection (Hermann, 1975) on T^*M . We show in this section that it corresponds to a generalized affine connection on the affine frame bundle AM, and that phase space with electromagnetic fields should accordingly be considered as the affine cotangent bundle AT^*M rather than T^*M. The reader is referred to the appendix for details relating to the affine frame bundle of a manifold and related associated bundles.

In the appendix we recall the fact that the bundles $E_1 = LM \times_{GL(4)} \mathbb{R}^{4*}$ and $E_2 = AM \times_{A(4)} \mathbb{R}^{4*}$ associated to LM and AM, respectively, are isomorphic with T^*M. The bundle E_2 is the **affine cotangent bundle**. If one is concerned only with the invariant representation of covectors on spacetime then T^*M is sufficient , and E_1 and E_2 need not be considered. However, in physical applications one wants to keep track of the components of covectors and the frames that define the components, and for such purposes the associated bundles E_1 and E_2 are especially useful. In particular, let us reconsider the theorem on affine connections on T^*M quoted in the last section.

The assumptions of the theorem require a coordinatization of T^*M by affine coordinates, and not simply linear coordinates. In the appendix it is shown (cf. equation A-13) that in order to introduce affine coordinates on T^*M one needs the identification of E_2 with T^*M; hence in order to discuss the geometry of affine connections on T^*M we also need the identification of E_2 with T^*M. If we consider the affine connection 1-forms θ_μ given in (29) as defined on $E_2 = AT^*M$, then we can compare this connection with affine connections on AM.

A generalized affine connection (Kobayashi and Nomizu, 1963) on AM is an $a(4) = gl(4) \oplus R^{4*}$ – valued 1-form ω satisfying the connection transformation law

$$R^*_{(a,\xi)}\omega = ad(a,\xi)^{-1} \cdot \omega \quad . \tag{32}$$

Here $R_{(a,\xi)}$ denotes right translation on AM by $(a,\xi) \in A(4)$ and ad denotes the adjoint action of A(4) on its Lie algebra $a(4)$. Let (U, x^μ) be a coordinate chart on M and t a covector field on U. Then the section $\sigma : U \rightarrow AM$ defined by

$$\sigma(p) = (p, \frac{\partial}{\partial x^\mu}|_p, t(p)) \tag{33}$$

is an affine frame field on M that defines affine coordinates (q^μ, π_ν) on AT^*M as in (10). The components ω_σ of ω relative to the affine frame field $p \rightarrow \sigma(p)$ are given by

$$\omega_\sigma := \sigma^*\omega = {}^\sigma\omega_L \oplus {}^\sigma\omega_T \quad . \tag{34}$$

The direct sum is in the Lie algebra $a(4)$, and the subscripts L and T refer to "linear" and "translational", respectively. If we denote the standard basis of $gl(4)$ by (E^μ_ν) and the standard basis of R^{4*} by (r^μ) then ω_σ may be expressed as

$$\omega_\sigma = \Gamma^\mu{}_\nu E^\nu_\mu \oplus {}^tK_\mu r^\mu \quad . \tag{35}$$

The $\Gamma^\mu{}_\nu = \Gamma^\mu_{\lambda\nu} dx^\lambda$ are linear connection 1-forms and are independent of the origin t of the affine frame, while the translational connection 1-forms ${}^tK_\mu = {}^tK_{\mu\nu} dx^\nu$ have left superscripts to indicate the dependence of the components of ω_T on the origin of the affine frame. If $p \rightarrow \bar{\sigma}(p) = (p, e_\mu, s)$ is another affine frame field on U then the components ${}^sK_\mu$ are related to the components ${}^tK_\mu$ by the R^{4*} connection transformation law (Hermann, 1975; Norris, 1985)

$${}^sK_{\mu\nu} = {}^tK_{\mu\nu} + \nabla_\nu(s_\mu - t_\mu) \quad . \tag{36}$$

Finally, from a variant of a theorem in differential geometry (Kobayashi and Nomizu, 1963) we know that affine connections on AM are in 1:1 correspondence with pairs $(\Gamma^\lambda_{\mu\nu}, {}^tK_{\mu\nu})$ defined as above.

To understand in a simple way how a connection on AM is related to the vector bundle affine connection given by (29) we compute the covariant derivative of a smooth section $\beta : M \rightarrow AT^*M$ of the affine cotangent bundle AT^*M. Associated with β is a unique function $f_\beta : AM \rightarrow A^{4*}$ defined by (Kobayashi and Nomizu, 1963)

$$f_\beta(p, e_\mu, t) := (p, e_\mu, t)^{-1}(\beta(p)) \quad . \tag{37}$$

From equations (10) and (A-13) we see that evaluating this function at an affine frame (p, e_μ, t) is equivalent to finding the coordinates of $(p, \beta(p)) \in AT^*M$ with respect to the affine coordinates (q^μ, π_ν) defined by the affine frame. We now compute the covariant derivative of β in two ways, first using ω on AM and then using the connection 1 forms θ_μ on AT^*M given by (29).

The exterior affine covariant derivative of β with respect to ω is defined in terms of f_β by

$$Df_\beta := df_\beta + \omega \cdot f_\beta \quad . \tag{38}$$

The "dot" in $\omega \cdot f_\beta$ denotes the action of the Lie algebra a(4) on A^{4*} induced by the standard action given in (A-7). Pulling the A^{4*} - valued 1-forms Df_β back to spacetime M using the section σ given in (33) and using (34) leads to

$$\begin{aligned} \sigma^*(Df_\beta) &= d(\sigma^* f_\beta) + {}^\sigma\omega_L \cdot (\sigma^* f_\beta) - {}^\sigma\omega_T \\ &= d(f_\beta \circ \sigma) + {}^\sigma\omega_L \cdot (f_\beta \circ \sigma) - {}^\sigma\omega_T \quad . \end{aligned} \tag{39}$$

We introduce the notation ${}^t\beta_\mu(p) = \beta_\mu(p) - t_\mu(p)$ for the coordinates of β with respect to the affine frame field (33). Then since

$$f_\beta \circ \sigma(p) = \pi_\mu(p, \beta) r^\mu = (\beta_\mu(p) t_\mu(p)) r^\mu \tag{40}$$

we may rewrite equation (39) in component form using (35) as

$$D^t\beta_\mu = d^t\beta_\mu - \Gamma^\nu{}_\mu({}^t\beta_\nu) - {}^tK_\mu \quad . \tag{41}$$

Evaluating these 1-forms at $\frac{\partial}{\partial x^\nu}$ yields the components formula

$$D_\nu({}^t\beta_\mu) = \partial_\nu({}^t\beta_\mu) - \Gamma^\lambda_{\nu\mu}({}^t\beta_\lambda) - {}^tK_{\nu\mu} \quad . \tag{42}$$

This is the local coordinate formula for the A(4) covariant derivative of an affine covector $\beta : M \to AT^*M$. The differences between this formula and the formula $\nabla_\mu\beta_\nu = \partial_\mu\beta_\nu - \Gamma^\lambda_{\mu\nu}\beta_\lambda$ for the linear covariant derivative of a covector $\beta : M \to T^*M$ are due to the extra R^{4*} degrees of freedom in AT^*M.

To evaluate the covariant derivative of $\beta : M \to AT^*M$ using the affine connection 1-forms θ_μ given in (29) we evaluate the pull back $\beta^*\theta_\mu$ of the θ_μ to M. The geometrical picture is that when a covector field is thought of as a section of AT^*M, then the image of its domain under the map $p \to (p, \beta(p))$ is a surface in AT^*M. The $\beta^*\theta_\mu$ are equivalent to the restrictions of the 1-forms θ_μ to (vectors tangent to) this surface. From (29) we find

$$\begin{aligned} \beta^*\theta_\mu &= \beta^*(d\pi_\mu) - \beta^*\omega_\mu - \beta^*(\omega^\nu{}_\mu \pi_\nu) \\ &= d(\pi_\mu \circ \beta) - \beta^*\omega_\mu - \beta^*\omega^\nu{}_\mu (\pi_\nu \circ \beta) \quad . \end{aligned} \tag{43}$$

Since

$$\begin{aligned} \pi_\mu \circ \beta(p) &= \pi_\mu(p, \beta(p)) \\ &= \beta_\mu(p) - t_\mu(p) \\ &= {}^t\beta_\mu \quad . \end{aligned} \tag{44}$$

we may rewrite (43) using (28) and (44) as

$$\beta^*\theta_\mu = d({}^t\beta_\mu) - \beta^*\omega^\nu{}_\mu({}^t\beta_\nu) - \beta^*\omega_\mu \quad . \tag{45}$$

The statement of the theorem quoted in Section 2 refers to a fixed, but arbitrary, affine coordinate system, and as such the notation used in the theorem is somewhat incomplete. The connection 1-forms ω_μ must also transform according to the rule (36). We will therefore write ${}^t\omega_\mu$ in place of ω_μ to indicate the dependence on choice of origin of the affine frame. According to the theorem of Section 2 the 1-forms $\omega^\nu{}_\mu$ and ${}^t\omega_\mu$ are pull backs under $proj_{AT}$ of 1-forms on M. Define 1-forms $\gamma^\nu{}_\mu$ and ${}^tk_\mu$ on M by

$$\omega^\nu{}_\mu := proj^*_{AT}(\gamma^\nu{}_\mu) \quad , \tag{46-a}$$

$${}^t\omega_\mu := proj^*_{AT}({}^tk_\mu) \quad . \tag{46-b}$$

Since $proj_{AT} \circ \beta = id \mid_M$, equation (45) can be reexpressed as

$$\beta^*\theta_\mu = d({}^t\beta_\mu) - \gamma^\nu{}_\mu({}^t\beta_\nu) - {}^tk_\mu \quad . \tag{47}$$

Comparing this equation with equation (41) leads to the identification

$$\Gamma^\nu{}_\mu = \gamma^\nu{}_\mu \tag{48-a}$$

$${}^tK_\mu = {}^tk_\mu \tag{48-b}$$

Since the pair $(\gamma^{\nu}{}_{\mu}, {}^{t}k_{\mu})$ defines an affine connection on $AT^{*}M$, and $(\Gamma^{\nu}{}_{\mu}, {}^{t}K_{\mu})$ defines an affine connection on AM, equations (48) gives a correspondence between affine connections on these two bundles.

In order to apply this correspondence to the affine connections (29)-(31) based on the charged symplectic form given in (21), we recall that (21) was defined relative to the non-canonical affine frame field $(p, e_{\mu}(p), 0)$. Thus we rewrite (29)-(31) as

$$ {}^{0}\theta_{\mu} = d\pi_{\mu} - {}^{0}\omega_{\mu} - \omega^{\nu}{}_{\mu}\pi_{\nu} \quad , \tag{49} $$

$$ {}^{0}\omega_{\mu} = \frac{e}{2}\hat{F}_{\mu\nu}dq^{\nu} \quad , \tag{50} $$

$$ \omega^{\nu}{}_{\mu} = 0 \quad . \tag{51} $$

From equations (48)-(51) we may infer the following result. The charged symplectic 2-form in special relativistic symplectic mechanics defines an affine connection on the affine frame bundle AM of flat spacetime. The Maxwell field tensor, thought of as the covector-valued 1-form $(F_{\mu\nu}dx^{\nu}) \otimes dx^{\mu}$, plays the role of the R^{4*} part of the connection, and the linear part is the flat Minkowski connection. Generalizing to a curved spacetime the linear part of the affine connection would correspond to the Riemannian linear connection, and in place of (51) one would find the $\omega^{\nu}{}_{\mu}$ being given by the 1-forms of the Levi-Civita connection.

4. Symplectic Structure from an Affine Connection on AM

In Sections 2 and 3 we saw how to use the charged symplectic 2-form on $AT^{*}M$ to define a generalized affine connection on the affine frame bundle of Minkowski spacetime. We now reverse the process and find the conditions that a generalized affine connection on AM define a symplectic structure on $AT^{*}M$.

Let ω denote a generalized affine connection on AM, with components $(\Gamma^{\mu}_{\nu}, {}^{t}K_{\lambda})$ given by (35) relative to the affine frame field (33). Consider the connection 1-forms ${}^{t}\theta_{\mu}$ defined on $AT^{*}M$ by

$$ {}^{t}\theta_{\mu} = d\pi_{\mu} - {}^{t}K_{\mu} - \Gamma^{\nu}{}_{\mu}\pi_{\nu} \quad . \tag{52} $$

To avoid excessive notation in the following formulas we are here abusing notation slightly by assuming that Γ and ${}^tK$ are defined on AT^*M. Define the linear and translational curvature 2-forms $\Omega^\nu{}_\mu$ and ${}^t\Phi_\mu$ by (Kobayashi and Nomizu, 1963; Hermann, 1975)

$$\Omega^\nu{}_\mu = \Omega_{\lambda\kappa\mu}{}^\nu dq^\lambda \wedge dq^\kappa = d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\xi \wedge \Gamma^\xi{}_\mu \tag{53}$$

$${}^t\Phi_\mu = {}^t\Phi_{\mu\lambda\kappa} dq^\lambda \wedge dq^\kappa = d({}^tK_\mu) - \Gamma^\nu{}_\mu \wedge {}^tK_\nu \quad . \tag{54}$$

The transformation law for the translational curvature 2-forms under the change of origin $(p, e_\mu, t) \rightarrow (p, e_\mu, t + \xi_\mu e^\mu)$ is

$${}^s\Phi_\mu = {}^t\Phi_\mu - \Omega^\nu{}_\mu \xi_\nu \quad . \tag{55}$$

The torsion 2-forms T^μ of the linear connection Γ may be expressed in the following various ways:

$$\begin{aligned} T^\mu &= T^\mu_{\lambda\ \kappa} dq^\lambda \wedge dq^\kappa \\ &= \Gamma^\mu_{[\lambda\kappa]} dq^\lambda \wedge dq^\kappa \\ &= \Gamma^\mu{}_\lambda \wedge dq^\lambda \quad . \end{aligned} \tag{56}$$

In Section 2 it was shown that the charged symplectic 2-form can be rewritten as $\tilde{S} = C_\mu \wedge dq^\mu$, and that $C_\mu = d\pi_\mu - \frac{e}{2} F_{\mu\nu} dq^\nu$ can be identified with the 1-forms of the translational part of a generalized affine connection on AM. Suppose now that we start with a generalized affine connection on AM and use it to induce 1-forms ${}^t\theta_\mu$ of a vector bundle affine connection on AT^*M. These 1-forms ${}^t\theta_\mu$ can then be used to define a 2-form $S := {}^t\theta_\mu \wedge dq^\mu$ on AT^*M. What are the necessary and sufficient conditions that the ${}^t\theta_\mu$ must satisfy in order that S be a symplectic form on AT^*M? Inserting the expressions given in (52) for the ${}^t\theta_\mu$ we obtain

$$\begin{aligned} S &= {}^t\theta_\mu \wedge dq^\mu \\ &= d\pi_\mu \wedge dq^\mu - {}^tK_{\mu\nu} dq^\mu dq^\nu - \Gamma^\nu_{\lambda\mu} dq^\lambda \wedge dq^\mu \pi_\nu \quad . \end{aligned} \tag{57}$$

Decompose ${}^tK_{\mu\nu}$ as

$${}^tK_{\mu\nu} = {}^tF_{\mu\nu} + {}^tH_{\mu\nu} \quad , \tag{58}$$

where

$${}^tF_{\mu\nu} := {}^tK_{[\mu\nu]} \tag{59 - a}$$

$$ {}^tH_{\mu\nu} := {}^tK_{(\mu\nu)} \quad . \tag{59-b} $$

Now (57) may be rewritten using (56), (58) and (59) as

$$ \begin{aligned} S &= d\pi_\mu \wedge dq^\mu - {}^tF_{\mu\nu} dq^\mu \wedge dq^\nu - T^\mu \pi_\mu \\ &= S_C - {}^tF - T^\mu \pi_\mu \quad . \end{aligned} \tag{60} $$

Thus only the antisymmetric part ${}^tF_{\mu\nu}$ of ${}^tK_{\mu\nu}$ and the torsion T^μ of Γ^ν_μ enter into the definition of S.

Theorem: The 2-form $S = {}^t\theta_\mu \wedge dq^\mu$ defines a symplectic 2-form on AT^*M if and only if (a) the associated linear connection is torsion free, and (b) $d({}^tF) = 0$.

Proof: Computing the exterior derivative of $S = {}^t\theta_\mu \wedge dq^\mu$ using (60) yields

$$ dS = -d({}^tF_{\mu\nu}) \wedge dq^\mu \wedge dq^\nu - (dT^\mu)\pi_\mu - T^\mu \wedge d\pi_\mu \quad . \tag{61} $$

Suppose that S is a symplectic 2-form so that both sides of (61) vanish identically. Then by linear independence the last term on the right hand side must vanish seperately since it is the only term that contains a factor of $d\pi_\mu$. This implies that

$$ T^\mu = 0 \quad , \tag{62} $$

so the associated linear connection is torsion free. The vanishing of the remaining terms on the right hand side of (61) now imply

$$ d({}^tF) = 0 \quad , \tag{63} $$

as was to be shown.

Conversely, suppose $S = {}^t\theta_\mu \wedge dq^\mu$ and the components of ${}^t\theta_\mu$ satisfy (62) and (63). Then from (61) we get $dS = 0$. The non-degeneracy of S follows from the structure of S and the ${}^t\theta_\mu$. ∎

Condition (b) of the theorem implies that locally ${}^tF_{\mu\nu} = {}^tK_{[\mu\nu]}$ is derivable from a potential:

$$ {}^tF_{\mu\nu} = \partial_\mu({}^tA_\nu) - \partial_\nu({}^tA_\mu) \quad . \tag{64} $$

Since the tensor ${}^tK$ that represents the translational part of the affine connection depends explicitly on the origin of the affine frame field, it may appear that $d({}^tF) = 0$ may hold with respect to one origin field t, but not with respect to another origin field. That this is not the case can be seen from the following argument.

It is well-known that the curvature 2-forms $\Omega^\mu{}_\nu$ of a torsion–free linear connection satisfy the identity

$$\Omega^\mu{}_\nu \wedge dq^\nu = \Omega^\mu_{\nu\lambda\kappa} dq^\nu \wedge dq^\lambda \wedge dq^\kappa = 0 \quad . \tag{65}$$

Using the transformation law (55) we can write

$${}^s\Phi_\mu \wedge dq^\mu = {}^t\Phi_\mu \wedge dq^\mu + \Omega^\nu{}_\mu \wedge dq^\mu \xi_\nu \quad . \tag{66}$$

Using (65) in this equation reduces it to

$${}^s\Phi_\mu \wedge dq^\mu = {}^t\Phi_\mu \wedge dq^\mu \quad . \tag{67}$$

This implies that the 3-forms ${}^t\Phi_\mu \wedge dq^\mu$ are actually independent of the origin of the affine frame when the associated linear connection is torsion free. Expressing these 3-forms in component form we find

$$\begin{aligned} {}^t\Phi_\mu \wedge dq^\mu &= d({}^tK_\mu) \wedge dq^\mu \\ &= \partial_{[\nu}{}^tK_{\mu\kappa]} dq^\nu \wedge dq^\kappa \wedge dq^\mu \\ &= \partial_{[\nu}{}^tF_{\mu\kappa]} dq^\nu \wedge dq^\kappa \wedge dq^\mu \quad . \end{aligned} \tag{68}$$

Thus

$${}^t\Phi_\mu \wedge dq^\mu = 0 \quad \Longleftrightarrow \quad d({}^tF_{\mu\nu} dq^\mu \wedge dq^\nu) = 0 \quad . \tag{69}$$

The result is that the condition that the 2-form ${}^tF_{\mu\nu} dq^\mu \wedge dq^\nu$, derived from ${}^tK_{\mu\nu} dq^\nu$, be **closed** is translational invariant. That is to say, if ${}^tF_{\mu\nu} dq^\mu \wedge dq^\nu$ is closed, then ${}^sF_{\mu\nu} dq^\mu \wedge dq^\nu$ is also closed. We formalize these remarks in the following

Corollary: If a generalized affine connection ${}^t\theta_\mu$ has a torsion–free linear part, then the condition $d({}^tF) = 0$ is independent of the origin of the affine frame.

5. P(4) Affine Reinterpretation of Canonical Equations

We have shown that the charged symplectic 2-form on AT^*M defines the R^{4*}–part of a generalized affine connection on the affine frame bundle AM of the spacetime manifold M. Conversely, if $(\Gamma^{\mu}{}_{\nu}, {}^tK_{\mu})$ is an affine connection such that (a) $\Gamma^{\mu}{}_{\nu}$ is torsion–free and (b) the 2-form ${}^tF$ derived from ${}^tK_{\mu}$ is closed, then we have a prescription for constructing a charged symplectic 2-form on AT^*M in which ${}^tF$ plays the role of the Maxwell field tensor. Thus that part of symplectic mechanics on AT^*M related to only the symplectic 2-form, that is independent of the choice of Hamiltonian, is related to the geometry of a class of generalized affine connections on the affine frame bundle of spacetime.

Now in symplectic mechanics one has a Hamiltonian in addition to the symplectic 2-form , and the corresponding canonical equations of motion. In the case we have been considering, specifically the charged symplectic form (21) together with the Hamiltonian (20), the equations of motion lead to the Lorentz force law on spacetime. Now that we know that the charged symplectic form induces a generalized affine connection on AM, we are led to ask for the **geometrical interpretation** of the canonical equations of motion relative to the induced affine connection. The result, which will not be very surprising to geometers and relativists, is that the Lorentz force law becomes the equation of a generalized affine geodesic with respect to a generalized affine connection. To show this we consider the equations of motion that follow from (15),(20) and (21). These equations can be put into the form

$$\dot{q}^{\mu} = \frac{\partial \mathcal{H}}{\partial \pi_{\mu}} \quad , \tag{70}$$

$$\frac{\bar{D}\pi_{\mu}}{Ds} = -\frac{\partial \mathcal{H}}{\partial q^{\mu}} \quad , \tag{71}$$

where

$$\frac{\bar{D}\pi_{\mu}}{Ds} := \frac{d\pi_{\mu}}{ds} - eF_{\mu\nu}\dot{q}^{\nu} \quad . \tag{72}$$

Remarks:

i. The notation $\frac{\bar{D}\pi_{\mu}}{Ds}$ used in (71) and (72) anticipates the result to be established below that (75) represents the covariant derivative, with respect to an affine connection, of an affine vector field π_{μ} along the trajectory of the particle.

ii. It is well–known that when the Hamiltonian is given by (20) the term

$$\frac{\partial \mathcal{H}}{\partial q^\mu} = \frac{1}{2m}\pi_\lambda \pi_\kappa \frac{\partial g^{\lambda\kappa}}{\partial q \mu}$$

on the right hand side of equation (71) brings in the Christoffel symbols of the metric linear connection. In the usual fashion these terms may first be reexpressed in terms of $\dot{q}^\mu$ using (70). The result can then be transferred to the left hand side of (71) and combined with $\frac{d\pi_\mu}{ds}$ to give $\nabla_{\dot{q}}\pi_\mu = \frac{d\pi_\mu}{ds} - \{{}^{\kappa}_{\mu\lambda}\}\pi_\kappa \dot{q}^\lambda$, the usual formula for the linear covariant derivative of a vector field along a curve.

iii. The coordinates (q^μ, π_ν) on AT^*M depend on the choice of affine frame. As discussed in Section 3 variables that depend on the choice of origin of the affine frame should carry an addition left superscript to remind us of this fact. Thus in particular we relabel our coordinates (q^μ, π_ν) as $(q^\mu, {}^t\pi_\nu)$, where t denotes the origin of the affine frame field defining these coordinates. As indicated in Section 3 the charged symplectic form given in the form (21) was defined relative to the affine frame field $(p, \frac{\partial}{\partial x^\mu}(p), 0)$. Accordingly we shall replace π_μ in our formulas with ${}^0\pi_\mu$.

When the steps outlined in these remarks are carried out equations (70)–(72) take the form

$$\dot{q}^\mu = \frac{1}{m} g^{\mu\nu}({}^0\pi_\nu) \quad , \tag{73}$$

$$\frac{D({}^0\pi_\mu)}{Ds} = 0 \quad , \tag{74}$$

where now

$$\frac{(D^0\pi_\mu)}{Ds} = \frac{d({}^0\pi_\mu)}{ds} - \{{}^{\kappa}_{\mu\lambda}\}({}^0\pi_\kappa)\dot{q}^\lambda - eF_{\mu\nu}\dot{q}^\nu \quad . \tag{75}$$

Having eliminated the Hamiltonian from the equations of motion we are now in a position to reinterpret equations (73)–(75) geometrically in terms of affine connections on AM. Recall first that these equations are defined with respect to the affine frame field $(p, \frac{\partial}{\partial x^\mu}(p), 0)$. Thus equation (73) defines the components of the affine vector field $\pi(s) = 0(\gamma(s)) \oplus m g_{\mu\nu}\frac{d\dot{\gamma}^\nu}{ds}$ along the trajectory $s \longrightarrow \gamma(s) = (x^\mu(s))$ of the particle (cf. the remark following equation (A-11)). Equations (74) and (75) then imply that the

Hamilton equations of motion (70)–(72) are equivalent to the equation of an affine geodesic (Hermann, 1975; Norris, 1985) with respect to the generalized affine connection

$$(\Gamma^{\mu}{}_{\nu}, {}^{0}K_{\mu}) = (\{^{\mu}_{\lambda\nu}\}dx^{\lambda}, -eF_{\mu\nu}dx^{\nu}) \quad . \tag{76}$$

Note that the translational part of this affine connection on AM is twice that of our earlier definition (50). We formalize our discussion in the following

Theorem: The equations of motion for the Hamiltonian $\mathcal{H} = \frac{1}{2m}(m^2 + g^{\mu\nu}\pi_\mu\pi_\nu)$ and charged symplectic form $S = d\pi_\mu \wedge dq^\mu + \frac{e}{2}F_{\mu\nu}dq^\mu dq^\nu$ are equivalent to the equations of an affine geodesic of the generalized affine connection given in (76).

Finally we wish to make a few remarks concerning the electromagnetic field equations. The discussion presented so far in this paper has been concerned with the equations of motion of charged particles in the presence of given external electromagnetic fields, and not with the Maxwell field equations of the electromagnetic field. However, since we now know that the Hamilton equations of motion in a curved Einstein–Maxwell spacetime are equivalent to the affine geodesic equations associated with the generalized affine connection $(\{^{\mu}_{\lambda\kappa}\}, -eF_{\mu\nu})$, we are led naturally to ask for the geometrical interpretation of the Maxwell field equations in terms of affine geometry. It has been shown (Norris, 1985) that the coupled Einstein–Maxwell field equations can be recast as geometrical equations stated in terms of the $P(4) = O(1,3) \otimes R^{4*}$ curvature of the affine connection $(\{^{\mu}_{\lambda\kappa}\}, -eF_{\mu\nu})$. Here we will consider only the Maxwell equations, and we refer the interested reader to an earlier paper (Norris, 1985) for a discussion of the full coupled Einstein–Maxwell equations.

Consider now a generalized affine connection (Γ, K) on AM such that in the affine gauge (local section of AM) $(p, e_\mu(p), o)$ the components of the connection are given by

$$\Gamma^{\mu}_{\lambda\kappa} = \{^{\mu}_{\lambda\kappa}\} \quad , \tag{77}$$

$$^{0}K_{\mu\lambda} = -F_{\mu\lambda} \quad . \tag{78}$$

Here $\{^{\mu}_{\lambda\kappa}\}$ denotes the Christoffel symbols of a spacetime metric tensor, and $F_{\mu\lambda}$ is an arbitrary antisymmetric type (0,2) tensor field on spacetime.

From equations (54) and (78) we have the following expression for the R^4–part of the affine curvature:

$$\begin{aligned} {}^0\Phi_{\mu\nu\lambda} &= \nabla_\nu {}^0K_{\mu\lambda} - \nabla_\lambda {}^0K_{\mu\nu} \\ &= \nabla_\lambda F_{\mu\nu} - \nabla_\nu F_{\mu\lambda} \quad . \end{aligned} \tag{79}$$

Theorem: The antisymmetric tensor field $F_{\mu\lambda}$ satisfies the source free Maxwell equations (a) $\nabla_{[\nu}F_{\mu\lambda]} = 0$ and (b) $\nabla_\mu F^\mu_\lambda = 0$ if and only if (c) ${}^0\Phi_{[\mu\nu\lambda]} = 0$ and (d) ${}^0\Phi^\mu_{.\mu\lambda} = 0$.

Proof: Suppose that $F_{\mu\lambda}$ satifies the Maxwell equations (a) and (b). Then by using the antisymmetry of $F_{\mu\lambda}$ it follows that (a) implies (c) and (b) implies (d). Conversely, if the R^4 curvature ${}^0\Phi_{\mu\nu\lambda}$ is constructed from the antisymmetric tensor $F_{\mu\lambda}$ as in (79), then it is easy to check that (c) implies (a) and (d) implies (b). ∎

The geometrical source–free Maxwell equations

$$\begin{aligned} {}^0\Phi_{[\mu\nu\lambda]} &= 0 \quad , \\ {}^0\Phi^\mu_{.\mu\lambda} &= 0 \end{aligned} \tag{80}$$

are thus analogous to the geometric source–free Einstein equations

$$R_{\mu\nu\lambda}{}^\mu = 0 \quad . \tag{81}$$

From the point of view of geometrical structure we may include the Riemannian zero-torsion identity $R_{[\mu\nu\lambda]}{}^\kappa = 0$ in the Einstein equations and rewrite equations (81) as

$$\begin{aligned} R_{[\mu\nu\lambda]}{}^\kappa &= 0 \\ R_{\mu\nu\lambda}{}^\mu &= 0 \quad . \end{aligned} \tag{82}$$

The parallel between the **affine Maxwell equations** (80) and the **Riemannian Einstein equations** (82) is now apparent.

6. Conclusions

The goal of this paper is to provide support for the $P(4) = O(1,3) \otimes \mathbb{R}^{4*}$ affine unified theory of gravitation and electromagnetism (Norris, 1985; Kheyfets and Norris, 1988). This support is provided by the link we have established between the charged symplectic form in standard Hamiltonian dynamics and affine connections on the affine frame bundle of spacetime.

When classical point particles are influenced by only the gravitational and electromagnetic fields, the classical equations of motion may be derived from the free-particle Hamiltonian and the charged symplectic form on the cotangent bundle T^*M of spacetime M. In Sections 2 and 3 of this paper we have shown that the charged symplectic form defines a generalized $P(4) = O(1,3) \otimes \mathbb{R}^{4*}$ affine connection on the affine frame bundle AM of spacetime. Turning things around in Section 4 we found the conditions that a generalized affine connection $(\Gamma^{\lambda}_{\mu\nu}, K_{\mu\nu})$ on AM must satisfy in order to define a symplectic form on T^*M. The conditions are that the associated GL(4) linear connection $\Gamma^{\lambda}_{\mu\nu}$ must be torsion free, and the skew-symmetric part $K_{[\mu\nu]}$ of the $\mathbb{R}^{4*}$ translational connection must locally take the form $K_{[\mu\nu]} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ for some local covector field A_μ.

In Section 5 we have shown that the classical equations of motion that follow from the free particle Hamiltonian and the charged symplectic form, that is the Lorentz force law, may be reinterpreted in affine geometry as the equation of a generalized affine geodesic with respect to the P(4) affine connection $(\{{}^{\mu}_{\lambda\kappa}\}, -eF_{\mu\nu})$ on the affine frame bundle AM of spacetime. In the affine picture the influence of the gravitational field on classical particles is the general relativistic interaction characterized by the Riemannian O(1,3) linear component $\{{}^{\mu}_{\lambda\kappa}\}$ of the generalized affine connection. The new feature in the affine picture is that the electromagnetic interaction is characterized by the translational part of the affine curvature, with the electromagnetic field tensor playing the role of the $\mathbb{R}^{4*}$ component $-eF_{\mu\nu}$ of the generalized affine connection.

In addition to this affine geometrization of the classical equations of motion of charged test particles in combined gravitational and electromagnetic fields, we also showed in Section 5 how the source-free Maxwell equations can be geometrized in terms of the $\mathbb{R}^{4*}$

affine curvature. The affine geometric Maxwell equations (80) are remarkably parallel in structure to the Riemannian Einstein equations (82). The full coupled Einstein-Maxwell equations have been reinterpreted elsewhere (Norris, 1985) in terms of a P(4) affine geometry, and recently a variational principle has been found (Chilton and Norris, 1990) that yields the P(4) affine Einstein-Maxwell equations.

APPENDIX.

We need the following facts and notations about the frame bundles of a 4-dimensional manifold (Kobayashi and Nomizu, 1963). The bundle of linear frames is the principal fiber bundle $LM \overset{proj_{LM}}{\longrightarrow} M$ whose points consist of pairs (p, e_μ) where (e_μ) is a linear frame at $p \in M$. The structure group Gl(4) acts on LM on the right by

$$(p, e_\mu) \cdot (a^\mu_\nu) \longrightarrow (p, \tilde{e}_\nu) = (p, e_\mu a^\mu_\nu) \quad \forall \ (a^\mu_\nu) \in Gl(4) \quad . \qquad (A-1)$$

The bundle of affine frames AM is the principal fiber bundle $AM \overset{proj_{AM}}{\longrightarrow} M$ whose points consist of triples (p, e_μ, t) where (e_μ) is a linear frame and t is a covector at $p \in M$. The structure group of AM is $A(4) = Gl(4) \otimes R^{4*}$, and its right action is as in (11-a) for $(a^\nu_\mu, \xi_\lambda) \in A(4)$. The bundle LM can now be identified with a subbundle L_0M of AM via the map $\gamma : LM \longrightarrow AM$ defined by

$$\gamma((p, e_\mu)) = (p, e_\mu, 0) \quad . \qquad (A-2)$$

All elements of AM on the same fiber with $(p, e_\mu, 0)$ can be obtained by the right action of A(4) on $(p, e_\mu, 0)$. Thus $(p, e_\mu, 0) \longrightarrow (p, e_\mu, 0) \cdot (a^\lambda_\nu, \xi_\nu) = (p, e_\lambda a^\lambda_\mu, \xi_\nu e^\nu)$, where (e^ν) is the coframe dual to (e_ν). These facts together with the map $\Pi : AM \to LM, \quad \Pi(p, e_\mu, t) = (p, e_\mu)$, defines AM as a trivial R^{4*} bundle over LM.

Given a vector space V on which Gl(4) acts on the left via a representation $\rho : Gl(4) \to Auto(V)$ one may use a standard construction (Kobayashi and Nomizu, 1963) to build the vector bundle $E \overset{proj_E}{\longrightarrow} M$ associated to LM, where $E = LM \times_{Gl(4)} V$. In particular, if $V = R^{4*}$ and $\rho(a^\lambda_\mu) = ((a^{-1})^\lambda_\mu))$, then $E_1 = LM \times_{Gl(4)} R^{4*}$ may be identified with T^*M as follows.

Each point of E_1 is an equivalence class $[(p, e_\mu), (\xi_\nu)]$, with $(p, e_\mu) \in LM$ and $(\xi_\nu) \in R^{4*}$. The equivalence is defined by the action of Gl(4) on $LM \times R^{4*}$ as follows: $(p, e_\mu, \xi_\nu) \sim (q, \bar{e}_\mu, \bar{\xi}_\nu) \Leftrightarrow p = q$ and $\exists (a^\mu_\nu) \in Gl(4)$ such that $\bar{e}_\mu = e_\nu a^\nu_\mu$ and $\bar{\xi}_\mu = a^\nu_\mu \xi_\nu$. The interpretation of the equivalence classes as covectors is very much in the classical vein. Select a representative pair $((p, e_\mu), (\xi_\nu))$ in an equivalence class and construct the covector

$\beta = \xi_\mu e^\mu$ at $p \in M$. Ever other member in the equivalence class consists of a frame $(p, \bar{e}_\mu)$ and the linear components of β relative to that frame. An equivalence class thus represents a covector at p by pairing off with each linear frame the frame components of the covector. E_1 and T^*M are therefore isomorphic under the map

$$[(p, e_\mu), (\xi_\nu)] \longrightarrow (p, \xi_\mu e^\mu) \quad , \qquad (A-3)$$

and we may consider each point $u = (p, e_\mu) \in LM$ as a linear map

$$u : (R^4)^* \longrightarrow (proj_{T^*M})^{-1}(proj_{LM}(u)) = T_p^*M \ , \qquad p = proj_{LM}(u) \quad . \qquad (A-4)$$

In the case of the cotangent bundle this map is given explicitly by

$$(p, e_\mu)(\xi_\lambda) = \xi_\mu e^\mu \qquad \forall (\xi_\mu) \in R^{4*} \quad , \qquad (A-5)$$

with inverse map

$$(p, e_\mu)^{-1}(\alpha) = (\alpha(e_\mu)) \qquad \forall\, \alpha \in T_p^*M \quad . \qquad (A-6)$$

We observe that equation (A-6) is equivalent to our first definition (2) of vertical coordinates π_μ on phase space T^*M for the free uncharged particle. To obtain the generalization (10) that includes the translations (9) we need to generalize $E(M, R^{4*})$.

Let A^{4*} denote R^{4*} with its natural affine structure (i.e. no preferred origin). Then A(4) acts on A^{4*} on the left by

$$(a^\mu_\nu, \xi_\lambda) \cdot (\beta_\kappa) = ((a^{-1})^\mu_\kappa (\beta_\mu - \xi_\mu)) \quad . \qquad (A-7)$$

Now consider the fiber bundle $E_2 = AM \times_{A(4)} A^{4*}$ associated to AM via the action (A-7). We will denote this bundle E_2 also by $AT^*M \overset{proj_{AT}}{\longrightarrow} M$ and refer to it as the **affine cotangent bundle**.

A point in E_2 is an equivalence class $[(p, e_\mu, t), (\xi_\nu)]$, and the interpretation is as follows. Select a representative pair $((p, e_\mu, t), (\xi_\nu))$ and define the covector

$$\beta = (\xi_\mu - t_\mu) e^\mu \quad . \qquad (A-8)$$

Similarly from another representative $((p, \bar{e}_\mu, \bar{t}), (\bar{\xi}_\nu))$ of the same equivalence class construct

$$\bar{\beta} = (\bar{\xi}_\mu - \bar{t}_\mu)\bar{e}^\mu \quad . \tag{A-9}$$

If $\bar{\beta} = \beta$ then

$$\bar{\xi}_\mu = a^\nu_\mu \xi_\nu + \eta_\mu \quad , \tag{A-10}$$

where $(a^\nu_\mu, \eta_\lambda)$ is the unique element of A(4) relating the two affine frames. An equivalence class in E_2 thus represents a covector β at $p \in M$ by pairing off each affine frame at p with the affine components of β with respect to the given frame. E_2 and T^*M are therefore isomorphic under the map

$$[(p, e_\mu, t), (\xi_\nu)] \longrightarrow (p, (\xi_\mu - t_\mu)e^\mu) \quad . \tag{A-11}$$

A convenient notation for a point β in E_2 is $\beta = t \oplus {}^t\beta$, where ${}^t\beta = \beta - t$ is the linear component of β with respect to the origin t.

Each point $w = (p, e_\mu, t) \in AM$ may now be considered as an affine map $w : A^{4*} \rightarrow AT^*_p M$, $p = proj_{_{AM}}(w)$, defined by

$$(p, e_\mu, t)(\xi_\nu) = (p, (\xi_\mu - t_\mu)e^\mu) \quad , \tag{A-12}$$

with inverse

$$(p, e_\mu, t)^{-1}(\beta) = \beta(e_\mu) - t_\mu \quad . \tag{A-13}$$

This mapping is equivalent to the definition (10) of the affine coordinates used for the charged particle system.

Further developments relating to the mathematical foundations of the affine geometry underlying the P(4) theory can be found in the paper by R. Fulp (Fulp, 1990).

REFERENCES

Chilton, J and Norris, L.K. (1990). *A Variational Principle for the P(4) Affine Theory of Gravitation and Electromagnetism*, to be published.

Fulp, R.O. (1990). *Affine Geometries Defined by Fiber Preserving Diffeomorphisms*, to appear in The Journal of Geometry and Physics.

Hermann, R. (1975). *Gauge Fields and Cartan-Ehresmann Connections*, Part A, Math Sci Press, Brookline.

Kheyfets, A. and Norris, L.K. (1988). International Journal of Theoretical Physics, **27**, 159-182.

Kobayashi, S. and Nomizu, K. (1963) *Foundations of Differential Geometry*, Volume I, Interscience, New York and London.

Norris, L.K. (1985). Physical Review D, **31**, 3090-3098.

Sniatycki, J. (1974). Journal of Mathematical Physics, **15**, 619-620.

Torrence, R.J. and Tulczyjew, W.M. (1973). Journal of Mathematical Physics, **14**, 1725-1732.

Woodhouse, N. (1980). *Geometric Quantization*, Clarendon Press, Oxford.